# Synthetic Cookware

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# Contents

## Preface

I originally intended that the opening line of this book should read,

> This book collects some utensils and implements needed for the computation of stable stems. A sensible reader should skip it, returning to the material when necessary.

a reference to Chiral Algebras. However, as the scope of the book expanded it became increasingly unclear where such an opening line should be placed. In its present form, the purpose of this book is fourfold,

(1) to give a self-contained introduction to the newest available machinery for understanding the Adams spectral sequence (synthetic spectra),
(2) to provide readers with as large a collection of tools and utensils as is reasonable,
(3) to provide experts with a single unified reference text,
(4) to demonstrate the utility of design, interfaces and abstraction barriers in mathematics (the synthetic viewpoint).

Although I have made an effort to make this book approachable for beginners, when balanced against constraints of space and the nature of the subject material I imagine the average reader to be a young researchers with a background in algebraic topology and a working knowledge of the material in [**HTT**] and [**HA**].

In order to explain the utility of synthetic spectra to understanding the Adams spectral sequence, let us consider three viewpoints on the Adams spectral sequence. In the first, the Adams spectral sequence is nothing more than a pile of homological algebra with a meaningful output. From this perspective a class on the $E_3$-page has no particular meaning and a typical argument about such a class might proceed by assuming this class to be permanent, using the associated map of spectra to make a contradiction, then concluding there must be a differential (and hopefully finding there is only one reasonable differential). The second viewpoint on the Adams spectral sequence is the "resolution viewpoint". Here the Adams spectral sequence is seen as a pile of spectra resolving some fixed object of study. From this perspective a class on the $E_3$-page has a specific meaning and it really does live somewhere. The argument by contradiction outlined above might now be replaced by a direct computation of the differential (and the dependence on the parenthetical removed). In trying to make this viewpoint systematic one is led to consider something like the $E^2$ model category of Dwyer–Kan–Stover [**DKS93**]. The fundamental weakness of this viewpoint is the human aspect; the underlying "pile of spectra" is large, unwieldy and carrying it around everywhere you go is mentally taxing.

Finally, we come to the "synthetic viewpoint". After studying Adams resolutions closely one finds a number of key formal properties these objects share and which can be used to interact with them. In the synthetic viewpoint on the Adams spectral sequence we forget the underlying "pile of spectra" from the resolution viewpoint, retaining the aforementioned "formal properties" as our only means of interacting with the object which we now rechristen *a synthetic spectrum*. At first it may seem that the synthetic viewpoint is less powerful than the resolution viewpoint. After all, it certainly is more restrictive. However, in this case *less is more*.

This brings us to a discussion of the fourth purpose of this book, that of design. In Chapter 2 we will identify what we believe to be the key properties of the Adams spectral sequence and construct an interface for interacting with this spectral sequence based on those properties alone. The strength of this interface lies in its simplicity. The constructions of [**Pst18**] now become implementation details—a proof that a category with certain formal properties exists. Through the remainder of this book we demonstrate that every known technique for computing Adams differentials becomes simpler synthetically and that many arguments previously regarded as heuristic can be made precise and rigorous with surprisingly little pain. Again we emphasize, everything that can be done synthetically *could* be done using resolutions directly, the fundamental advancement is one of clarity.

This books consists of six chapters, with the core material being contained in chapters 3-6. Chapter 1 is introductory and builds an interface for interacting with the titular *presentably symmetric monoidal categories*. Chapter 2 provides an overview and introduction to synthetic spectra with many proofs deferred to next chapter. Essentially all the of the material in this chapter also appears in [**Pst18**] and it is included for the purpose of self-containedness. Chapter 3 is on constructions. This has two components external constructions (i.e. constructing new categories analogous to synthetic spectra) and internal constructions (i.e. constructing new objects of interest inside a fixed synthetic category). Chapter 4 is on products, in it we study products and Toda brackets internal to a fixed synthetic category. For the reader interested in the computation of stable stems this chapter and chapter 6 will be of the greatest interest. In Chapter 5 we study power operations in the synthetic context and recast several classical theorems in these terms. In Chapter 6 we return to our roots and apply the material from the previous chapters to the study of the Adams and Adams–Novikov spectral sequences.

## Conventions

In order to maintain a more conversational style we establish several conventions which will be in force throughout the text.

**Categories.** Throughout this work the word category will refer to an $\infty$-category in the sense of Lurie, as set out in [**HTT**] and [**HA**] [1]. Although in these works Lurie uses a fixed model for $\infty$-categories, that of quasi-categories, our work will be model independent in the sense that we will purposefully only use those operations and results which we believe should be common to any "good theory of $\infty$-categories".

Amplifying the previous paragraph, there is an abstraction barrier between the implementation of $\infty$-categories (a pile of combinatorics) and the theory of $\infty$-categories (an abstract framework) and we will work exclusively with the latter. The benefits of this arrangement are not so much mathematical as organizational, allowing us to write more concisely.

**Articles and equivalence.** A key feature of working $\infty$-categorically is that it is essentially impossible to distinguish between something which has been specified uniquely and something which has been specified "up to contractible choice". Dually, it is essentially impossible to provide greater specificity in constructing an object than "up to contractible choice". This leads to two points which deserve some clarification, the first is linguistic, the second is notational.

English has two articles *a* and *the* and the distinction between their use is often whether an object has been uniquely specified. With the shift in our notion of "uniqueness" our use of articles also shifts. In order to assist the reader we offer the following heuristics for resolving the precise meaning of our language.

[1] This does not lead to issues because the $(2,1)$-category of 1-categories is a full subcategory of the $\infty$-category of $\infty$-categories.

- Typically one would extract from a noun phrase without a determiner "the collection of objects which match the noun phrase", instead one should extract "the[2] space of objects which match the noun phrase".
- The indefinite article *a* refers to having selected an (arbitrary) point in the space provided by the associated noun phrase.
- The definite article *the* asserts that the space provided by the associated noun phrase is contractible.

Example. Let us examine the two phrases "a sphere" and "the $n$-sphere". The first is relatively simple and might refer to any object equivalent to some fixed reference sphere. The second is more sticky. Based on the use of the definite article it is clear that an object is meant to be specified uniquely. Starting in dimension zero, the space $S^0$ is uniquely specified by the property that it is the unit for the smash product of spaces. The $n$-sphere can now be uniquely specified as the $n$-fold suspension of the 0-sphere[3]. The difference between "a sphere" and "the sphere" is the progenitor of all signs in algebraic topology.

Example. Although quasi-categories do not come equipped with a composition operation, suppose we have a pair of composable arrows $f$ and $g$ then by [**HTT**, Corollary 2.3.2.2] the space of compositions of $f$ and $g$ is contractible. Thus, we may speak of *the composition* of $f$ and $g$.

An emergent phenomenon in the $\infty$-categorical world is that notion of equivalence, which replaces equality, is substantially more subtle. For this reason we use four different symbols for equivalence.

- We will use $:=$ when two objects are definitionally equivalent.
- We will use $=$ when there is a unique equivalence between two objects, so that we may unambiguously refer to "the equivalence" (the space of equivalences between the pair of objects is contractible).
- We will use $\cong$ when an equivalence is given[4].
- We will use $\simeq$ when two objects are equivalent, but a choice of equivalence is not specified.

**Localization.** At almost every point in this work we will have a fixed prime $p$ in mind at which we will have localized all objects.

## Notation

Across the text we introduce several terms and abbreviations.

- A *psmc* (pl. psmc) is a presentable, symmetric monoidal category whose tensor product commutes with colimits separately in each variable.
- *sseq* is an abbreviation for spectral sequence.

We will typically use $\otimes$ for the monoidal structure on a psmc. In particular, we will use $\otimes$ for the smash product of spectra. Similarly, in a stable category we will use $\oplus$ for finite coproducts (and finite products).

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[2] It's turtles all the way down.

[3] The reader might complain that this is too involved for the phrase "the $n$-sphere" to be used without explanation, the author agrees, hence the inclusion of this example.

[4] Often our use of $\cong$ in a theorem statement will depend on a construction only given the body of the proof.

| | | |
|---|---|---|
| $\mathcal{S}$ | the category of spaces | |
| Sp | the category of spectra | |
| $\mathbb{S}$ | the sphere spectrum | |
| MU | complex cobordism | |
| BP | the Brown–Peterson spectrum | |
| $\mathbb{Z}$ | the integral Eilenberg–MacLane spectrum | |
| $\mathbb{F}_p$ | the mod $p$ Eilenberg–MacLane spectrum | |
| $\mathrm{Map}(-,-)$ | mapping space in a category | |
| $[-,-]$ | homotopy classes of maps | $\pi_0 \mathrm{Map}(-,-)$ |
| $\mathrm{map}(-,-)$ | mapping spectrum in a stable category | |
| $\mathrm{Syn}_E$ | $E$-synthetic spectra | [**Pst18**, Definition 4.1] |
| $\mathbb{S}^{k,s}$ | a bigraded sphere (see **??**) | |
| $\Sigma^{k,s}$ | a bigraded suspension | $\mathbb{S}^{k,s} \otimes -$ |

## Acknowledgments

The material in this book was developed over several years and the individual pieces were discussed and refined through conversation with the members of the MIT/Harvard topology community long before they were put to print. Among these, each of Tom Bachmann, Dexter Chua, Jeremy Hahn, Piotr Pstragowski, Andy Senger and Zhouli Xu have had a substantial impact on my understanding of the subject and have imparted their own unique influence onto the final product displayed here. The author would also like to thank Shaul Barkan, Mark Behrens, Shachar Carmeli, Sanath Devalapurkar, Mike Hopkins, Dan Isaksen, Cameron Krulewski, Ishan Levy, Haynes Miller, Doug Ravenel and Tomer Schlank for helpful conversations.

We offer our greatest thanks to Andy Senger. Ultimately it was his continued interest and encouragement that played the greatest role in the completion of this book.

## A note to readers of this version (v0.2)

As the reader is likely well aware, this is an incomplete version of this text. It has been made available on the authors' website on the basis that something is probably better than nothing. Many internal references between sections are broken in this version. Although the numbering of the chapters is stable, section and subsection numbering is likely to change.If you're interested in the content of a specific section (written or unwritten) feel free to email me. I'm always happy to discuss this material.

(v0.2) The present version is quite messy and we warn any readers that this version exists for the explicit purpose of making minimal versions of certain claims available for my own use elsewhere. Much of the core material in chapter 4 is in place. Chapter 6 is much more skeletal.

CHAPTER 1

# psmc

The primary characters in this work are *stable presentably symmetric monoidal categories*, their objects and the morphisms between those objects. In this chapter we set out these players and the devices we will use to interact with them. This does not mean that we will provide proofs (or even complete definitions). There is no replacement for a close reading of [**HTT**], [**HA**] and [**SAG**] and what we provide here is an interface for our own later use. For the expert this chapter may be superflous, for the novice it may be unreadably brief, but we hope that for those between these extremes it offers a welcome "user's guide".

This chapter covers five main topics: finiteness conditions, algebraic constructions, positivity, measurement and graphical calculi.

## 1.1. Finiteness conditions

**Definition 1.1.1.** We introduce the following finiteness conditions which might be imposed on an object $X$ in a psmc $\mathcal{C}$:

(1) $X$ is *compact* if $\mathrm{Map}(X, -)$ commutes with filtered colimits.
(2) $X$ is *projective* if $\mathrm{Map}(X, -)$ commutes with all colimits.
(3) $X$ is *dualizable* if
(4) $X$ is *invertible* if there exists a $Y$ such that $X \otimes Y \simeq \mathbb{1}$.

Note that if $\mathcal{C}$ is stable, then $\mathrm{Map}(X, -)$ commutes with finite colimits automatically, so it will suffices

**Definition 1.1.2.** (1) presentable,
(2) compactly generated,
(3) rigidly generated,
(4) picard generated,
(5) monogenic,
(6) unit monogenic

**Example 1.1.3.** The category of spectra, Sp is unit monogenic.

**Example 1.1.4.** $\mathrm{Sp}^{C_2}$ is picard generated.

**Example 1.1.5.** $\mathrm{Sp}^{C_p}$ is rigidly generated and if $p \neq 2$ it is not picard generated.

**Definition 1.1.6.** locally compact,

## 1.2. Algebraic constructions

### 1.2.1. Affineness.

### 1.2.2. Localizations.

1.2.2.1. *Bousfield localization.*

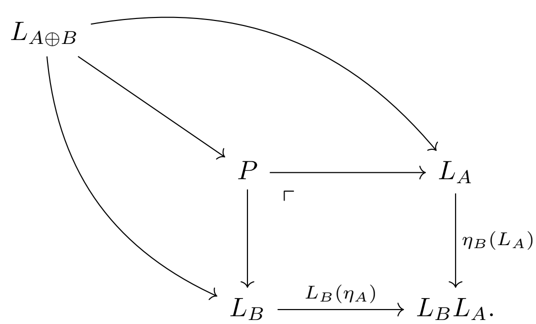
1.2.2.2. *Fracture squares.*

Up to this point we have been happy to consider a single localizations at a time. However, in practice one often needs to understand the relation between several different localizations. Here we introduce the main technique for doing this: fracture sqaures. The history of fracture squares is somewhat muddled, though it seems likely that they were known to Bousfield possibly as early as the late 70's. In our presentation we follow Bauer's note on the subject from [**DFHH14**, chapter 6].

**Lemma 1.2.1.** *Given objects $A$ and $B$ in a stable psmc $\mathcal{C}$ there are associated Bousfield localizations $L_A$, $L_B$ and $L_{A\oplus B}$. If we suppose $L_B$ preserves $A$-equivalences*[1]*, then there is a pullback square of symmetric monoidal functors*

$$\begin{array}{ccc} L_{A\oplus B} & \longrightarrow & L_A \\ \downarrow & \ulcorner & \downarrow \\ L_B & \longrightarrow & L_A L_B. \end{array}$$

Proof. In order to prove this lemma we just need to show that the diagonal map in the diagram below is an equivalence.




It suffices to show the the natural map $\mathrm{Id} \to P$ is an $(A \oplus B)$-equivalence. The map $\eta_B(L_A)$ is a $B$-equivalence, therefore $P \to L_B$ is a $B$-equivalence. Now since $\mathrm{Id} \to L_B$ is a $B$-equivalence we learn that $\mathrm{Id} \to P$ is a $B$-equivalence by 2-out-of-3.

The map $L_B(\eta_A)$ is an $A$-equivalence since $\eta_A$ is an $A$-equivalence and $L_B$ preserves $A$-equivalences by hypothesis. This implies that $P \to L_A$ is an $A$-equivalence and using 2-out-of-3 again we may conclude that $\mathrm{Id} \to P$ is an $A$-equivalence as well. $\square$

### 1.2.3. Short exact sequences.

## 1.3. Positivity conditions

The basic example of positivity (in the sense we use that word here) is an upper triangular matrix. In this section we discuss categorical analogs of upper triangularity and a number of situations in which this occurs automatically.

### 1.3.1. $t$-structures.

### 1.3.2. $w$-structures.

[1] Said another way, we're asking that $A \otimes X = 0$ implies $A \otimes L_B X = 0$.

## 1.4. Measurement

Up until this point most of our discussion has been relative. This is useful when the goal is to show a pair of objects are equivalent, but less so when the goal is to "understand the object $X$". In this section we turn to the problem of measurement. In the category of spectra there are several standard ways of a measuring an object so we begin by reviewing this example.

## 1.5. A key example: filtered spectra

CHAPTER 2

# Synthetic spectra

For moment we leave here only the following warning. More will appear in later versions.

**Warning 2.0.1.** We take the convention that the bigraded sphere $\mathbb{S}^{k,s}$ is $\Sigma^{-s}\nu\,\mathbb{S}^{k+s}$ and $\Sigma^{k,s}- := \mathbb{S}^{k,s}\otimes -$. This has the pleasant feature that the indices $(k, s)$ correspond to the $(x, y)$ coordinates in an Adams chart. Unfortunately this breaks with the preceeding literature.

CHAPTER 3

# Constructions

In this section we will become acquainted with some the basic objects and constructions in synthetic spectra. In the first section we will examine the constructions that can be made using the map $\tau$, paying particular attention to the various $\tau$-Bocksteins. This material will be used extensively throughout the rest of the paper. In the second section we give an application of $\tau$-Bocksteins, providing a synthetic reinterpretation of Toda's obstruction theory for Steenrod comodules. In the third and fourth sections we will study several natural truncation and weight structures on the category of synthetic spectra. Although it will be critical in several proofs [which?] reading this section is not strictly necessary to understand later material.

## 3.1. New categories from old

## 3.2. Categorical patterns

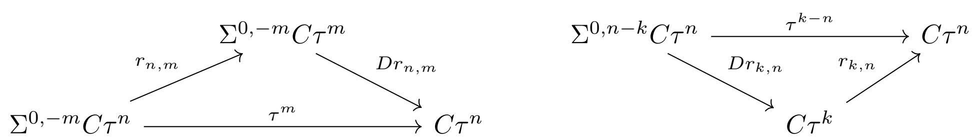
## 3.3. $\tau$-Bocksteins

Each synthetic spectrum is equipped with a cannonical endomorphism $\tau$. As we discussed in **??** we think of $\tau$ as part of the data recording the Adams filtration.

**Construction 3.3.1.** For any $n \geq m$ we may construct a cofiber sequence

$$\Sigma^{0,-m} C\tau^{n-m} \xrightarrow{\mathbb{D}r_{n,n-m}} C\tau^n \xrightarrow{r_{n,m}} C\tau^m \xrightarrow{\delta_{n,m}} \Sigma^{1,-m-1} C\tau^{n-m}$$

The identification of the leftmost map as $\mathbb{D}r_{n,n-m}$ will appear in **??**. We also have the following commuting diagrams and compatibilities between these maps:




- $r_{a,b} r_{b,c} = r_{a,c}$.

The last two triangles are not special to $\tau$, they're just properties of taking the cofiber by a self-map.

The notation established here is cumbersome, I'm going to change it before this is public

**Claim 3.3.2.** *$\pi_{**}(C\tau) \cong E_2^{**}(\mathbb{S})$ and $\delta_{**}$ computes the Adams differentials.*

**Claim 3.3.3.** *The natural map*

$$C\tau^{2n} \to C\tau^n$$

*is a sqaure-zero extension.*

Much latter this claim will inform several choices. This says that if you know everything about the $E_{n+1}$ page, then all the differentials needed to jump to the $E_{2n+1}$ page are linear in an appropriate sense.

**Proposition 3.3.4.** *The map on bigraded homotopy groups induced by*

$$\delta_{2n,n} : C\tau^n \to \Sigma^{1,-n-1} C\tau^n$$

*is a derivation.*

Proof. Since $C\tau^n$ is base-changed from $C\tau^n$ in the filtered setting it follows from (ref) that if we think of $(C\tau^n)^{\otimes 2}$ as a commutative $C\tau^n$-algebra via the left unit, then it is a trivial square-zero extension of $C\tau^n$ by $\Sigma^{1,-n-1}C\tau^n$. We write $\epsilon$ for a choice of generator of this square-zero ideal. From the pullback square

$$\begin{array}{ccc} C\tau^{2n} & \longrightarrow & C\tau^n \\ \downarrow & & \downarrow \eta_L \\ C\tau^n & \xrightarrow{\eta_R} & (C\tau^n)^{\otimes 2} \end{array}$$

We can then read off that the map $\delta_{2n,n}$ is the $\epsilon$-component of $\eta_R$. Now in order to conclude we can make the following calculation:

$$\begin{aligned} xy + \epsilon\delta_{2n,n}(xy) &= \eta_R(xy) = \eta_R(x)\eta_R(y) \\ &= (x + \epsilon\delta_{2n,n}(x))(y + \epsilon\delta_{2n,n}(y)) \\ &= xy + \epsilon(\delta_{2n,n}(x)y + (-1)^{|x|}x\delta_{2n,n}(y)) \end{aligned}$$

□

Due to the usefulness of the Liebniz rule we take some time to explain how to use it in some messier settings which tend to arise in nature. Suppose we are given $x \in \pi_{**}C\tau^n$ and $y \in \pi_{**}C\tau^m$ with $n \geq m$. The natural way to apply the Liebniz rule would be to first project $x$ down to $C\tau^m$ using $r$ and then consider $\delta$ applied to the product $r(x)y$. Carrying out this computation using the above proposition we learn that

$$\begin{aligned} \delta_{2m,m}(r_{n,m}(x)y) &= \delta_{2m,m}(r_{n,m}(x))y + (-1)^{|x|}r_{n,m}(x)\delta_{2m,m}(y) \\ &= \tau^{n-m}\delta_{n+m,n}(x)y + (-1)^{|x|}r_{n,m}(x)\delta_{2m,m}(y) \end{aligned}$$

Note that the output lives in $C\tau^m$ which may not be particularly useful if $n \gg m$. A finer result can be obtained if we instead use the dual of $r$ to push $y$ into $C\tau^n$ and take the product there,

$$\begin{aligned} \delta_{2n,n}(x(\mathbb{D}r_{n,m})(y)) &= \delta_{2n,n}(x)(\mathbb{D}r_{n,m})(y) + (-1)^{|x|}x\delta_{2n,n}(\mathbb{D}r_{n,m})(y) \\ &= \delta_{2n,n}(x)(\mathbb{D}r_{n,m})(y) + (-1)^{|x|}x\delta_{n+m,m}(y) \end{aligned}$$

Include all the cases where one of the numbers is infinity.

## 3.4. Toda's obstruction theory

## 3.5. Positivity

CHAPTER 4

# Products

In this chapter we explore the product structure on the bigraded synthetic homotopy groups of the sphere and composition operations more generally. The use of composition operations in homotopy theory is generally traced to Toda's determination of the first 19 stable stems in [**Tod62**]. In retrospect, Toda's ability to progress this far using such a circumscribed toolkit speaks to the power of these techniques.

Much of this section is ultimately a validation of Mark Mahowald's deep computational insight into the behavior of Toda brackets and differentials in the Adams spectral sequence. In their work on stable stems at the prime 2 [**IWX20**], Isaksen, Wang and Xu recognized the utility of a theorem of Moss which, under favorable circumstances, described the behavior of 3-folds in the Adams spectral sequence [**Mos70**]. The author thanks Zhouli Xu for explaining these ideas to him and suggesting that various extensions ought to exist.

In the first section we study the product structure on the brigraded homotopy groups of the sphere. The bulk of this section is devoted to extending the notion of a "hidden extension" to this setting. This is another step in the verification of the meta-claim that all information present in the Adams spectral sequence is contained in the synthetic homotopy groups. In the second section we give an introduction to composition operations. This material is mainly included for the sake of completeness. The notion of a Red-Blue brackets is new and is designed to provide a managable extension of Matric brackets to the "mixed-length" case. The author has found such brackets useful in practice and tools for manipulating them under-developed. In the third section we discuss methods of evaluating brackets which are specific to the synthetic category. These methods center on techniques for algebraically extracting the value of a bracket after tensoring with the cofiber of $\tau$. In the fourth section we study the $\tau$-bockstein and the consequences of its derivation structure. This material will be used in [**?**] where its ability to stretch known Adams differentials to obtain unknown ones is striking. In the final section of the chapter we carefully study the signs which arise in our constructions. At the prime 2 this question collapses, while at odd primes it has been a perennial thorn in our side.

This chapter is distinct from chapter 5 in that here we (mostly) explore the structure inherent to the stable category of synthetic spectra, whereas in chapter 5 we study the consequences of the symmetric monoidal structure.

## 4.1. The ring of homotopy groups

## 4.2. Brackets

Having discussed the product structure on synthetic homotopy groups the natural next step is a discussion of Toda brackets (and their generalizations) in the category of synthetic spectra. One of the most attractive features of Toda brackets in the synthetic category is the way in which they naturally capture the Adams filtration of Toda brackets in the ordinary category of spectra. Naively, one would expect that a Toda bracket $\langle \alpha, \beta, \gamma \rangle$ has filtration $l+m+n-1$ where $l, m, n$ are the filtrations of the three maps respectively, since this is what happens with Massey products. We will find that two more terms must be added to this sum which account for the Adams filtration of the nullhomotopy of $\alpha\beta$ and the nullhomotopy of $\beta\gamma$ respectively.

We begin this section with a discussion of brackets in a general (monoidal) stable category $\mathcal{C}$. This is meant to provide background and seperate out the parts that come from general theory. In the category of spectra we seperate out a certain class of 3-fold brackets which experience has shown are particularly useful. Among these brackets are included $v_n$-multiplication operations from which most known infinite families of elements are constructed. Next we discuss manipulations of brackets such as shuffling and Jacobi identites, again in a general category $\mathcal{C}$. In our discussion of shuffling we introduce new notation which allows us to keep track of auxillary information about nullhomotopies.

Specializing the preceding discussion to the case of synthetic spectra we recover theorems of [who?] on the relation between Massey products and Toda brackets. A highlight of this reformulation is a precise understanding of the notion of crossing differential which appears as a technical condition in several previous theorems. As an application of the techniques we develop we prove a Moss' theorem for 4-folds which allows one to evaluate 4-folds formed using Adams differential of mixed length. In the final subsection of this section we devote particular attention to the Massey product $M$ of [cite] and its connection to multiplication by $\theta_{4.5}$.

### 4.2.1. An introduction to brackets.

**Definition 4.2.1.** First we treat the case of 3-fold brackets.

If we admit brackets of the third kind as our fundamental operation, then all orders of bracket can be described in terms of two fundamental operations:

- **Composition:** given two composable maps $f : X \to Y$ and $g : Y \to Z$ we may form the composite $fg$.
- **Lifting:** given two composable maps $f$ and $g$ and a nullhomotopy $\gamma$ of the composite we may form maps $\hat{g} : cof(f) \to Z$ and $\hat{f} : X \to fib(g)$.

**Definition 4.2.2.** In a quasicategory foundation of $\infty$-categories an $n$-fold Toda bracket in a pointed cateogry $\mathcal{C}$ consists of a diagram of shape $\partial\Delta^n$ such that each 1-simplex $[i, i+k]$ is sent to the zero map for $n > k \geq 2$. The value of the bracket is extracted as follows: A map $F : \partial\Delta^n \to \mathcal{C}$ is the same data as a map $S^{n-2} \to \mathrm{Map}(F(0), F(n))$. Assuming $\mathcal{C}$ has colimits this is the data as a map element of $\mathrm{Map}(\Sigma^{n-2}F(0), F(n))$.

For example, a threefold $\langle f, g, h\rangle$ is recorded as follows: The 1-simplex $[0, 1]$ is $f$, the 1-simplex $[1, 2]$ is $g$ and the 1-simplex $[2, 3]$ is $h$. The 2-simplex $[0, 1, 2]$ is the nullhomotopy of $fg$, the 2-simplex $[1, 2, 3]$ is the nullhomotopy of $gh$. The value of the bracket is the associated circle in $\mathrm{Map}(0, 3)$.

FILTERED OBJECT VERSION

**Definition 4.2.3.** A Toda bracket

OBSTRUCTION TO BUILDING A PULLBACK
leave it as an exercise to equate all these.
depth

**4.2.2. Example: Red-Blue brackets.** As an example we study a class of 3-folds in Sp which we have dubbed red-blue brackets. These 3-folds have input and output spheres, but are more general than the Matric brackets considered elsewhere. We have found that in practice this collection of brackets is both simple enough to be analyzed in many cases and also rich enough to include many of the important phenomenological aspects of the stable stems.

**Definition 4.2.4.** Given a finite complex $X$ with specified bottom and top cells we may form the the bracket $X(-)$ which is described as follows. Let $Y$ denote the complex given by removing the top and bottom cell of $X$ and let $f$ and $g$ denote the attaching maps for these cells respectively (we also have a specific nullhomotopy of $fg$ as part of the data), then $X(-)$ is defined to be $\langle f, g, -\rangle$.

**Remark 4.2.5.** The name red-blue bracket comes from a picture of a cell structure for $X$ in which its bottom cell is colored blue and its top cell is colored red.

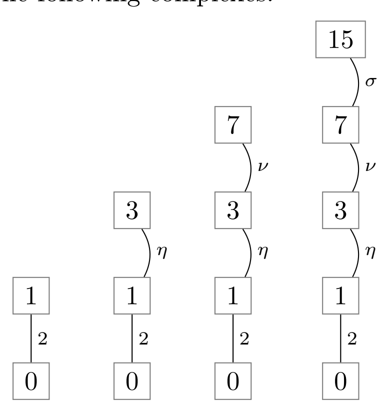
In practice we will also ask for the following further conditions: $X$ is finite, has a single bottom cell in degree 0, has a single top cell in degree $n$, and all other cells lie in degrees $[1, n-1]$.

**Example 4.2.6.** The degenerate case where $X$ is a two cell complex corresponds to multiplication by the attaching map. The simplest non-degenerate case (when $X$ is a 3-cell complex), corresponds to a standard 3-fold bracket.

**Example 4.2.7.** If the 3-cell complex constructed from a nullhomotopy of $2\sigma \circ 8$ is referred to as $P$, then $P(-) := \langle 2\sigma, 8, -\rangle$. This agrees with the somewhat standard use of $P$ for the Adams periodicity operator. If the 3-cell complex constructed from a nullhomotopy of $\bar{\kappa}_2 \circ 8 = 0$ is referred to as $M$, then $M(-) := \langle \bar{\kappa}_2, 8, -\rangle$. This agrees with the homotopical version of the Massey product operator $M$ [refs].[1]

**Example 4.2.8.** The spectrum $Y := cof(2) \otimes cof(\eta)$ provides a bracket $Y(-)$ which is defined whenever $2x = 0$ and $\eta x = 0$. In our discussion in [location] we will connect the operation $Y(-)$ to difficulties in completely calculating the $\mathbb{F}_2$-Adams spectral sequence above a line of slope 1/5. Similar difficulties do not arise at odd primes.

**Example 4.2.9.** Consider the following complexes:




[1]These two examples are what suggested the notation $X(-)$ to the author.

which we refer to as $Q_0, Q_1, Q_2$ and $Q_3$ respectively.[2] Since $Q_n$ (the homology operation) detects the shortest length differentials in the AHSS converging to $k(n)_*(-)$ we may think of the brackets $Q_n(-)$ as inducing multiplication by $v_n$.

As a follow up to this example we will codify what it means for the bracket based on a complex $X$ to induce multiplication by $v_n$.

**Definition 4.2.10.** Let $X$ be a red-blue complex which satisfies the extra requirements of remark 4.2.5 such that in the Atiyah-Hirzebruch spectral sequence computing the $k(n)$-homology of $X$ there is a differential $d(\text{red}) = v_n^N(\text{blue})$. Then, we refer to $X(-)$ as a $v_n^N$*-multiplication.*

This definition is based on the fact that the attaching map for the red cell is an inclusion on $k(n)$-homology which hits $v_n^N$ times the generator corresponding to the blue cell.

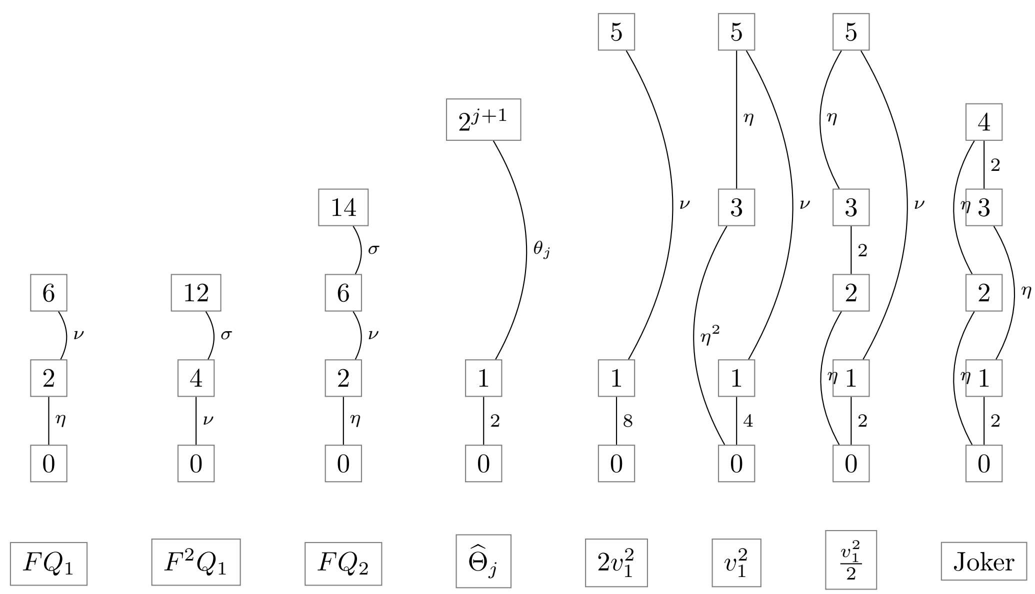
It is well known that a finite complex only admits a $v_n$-self map if its $k(n-1)$-homology is zero. Further, every nonzero finite complex whose $k(n-1)$-homology is zero has $\mathbb{F}_p$-homology of rank at least $2^n$. This provides a quantitative sense in which multiplication by $v_n$ is "difficult to define on the sphere". Based on this we prove the following lemma which duplicates the above claim in terms of red-blue brackets.

**Lemma 4.2.11.** *Any red-blue complex $X$ whose bracket is a $v_n^N$ multiplication in the sense of definition 4.2.10 has at least $n+1$ cells.*

write proof

PROOF. Look at BP homology. □

**Example 4.2.12.** The following are some commonly encounterd red-blue brackets. The bottom row gives each complex a name.[3]

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[2] The reasons for the naming are variously, $Q_1$ looks like a question mark and in homology the operation $Q_i$ takes the bottom cell of $Q_i$ to its top cell.

[3] F stands for frobenius.

**4.2.3. Shuffling brackets.**

**Lemma 4.2.13.** *If $fg = gh = hj = 0$, then*

$$\langle f, g, h\rangle j \cap f\langle g, h, j\rangle \neq \emptyset.$$

**Proposition 4.2.14.** *The following shuffling identities hold whenever all quantities are defined*

- *Given $gh \sim_\epsilon 0$, $hj \sim_{\epsilon'} 0$,*
$$f\langle g, h, j\rangle = \langle fg, h, j\rangle$$
- *Given $fg \sim_\epsilon 0$, $gh \sim_{\epsilon'} 0$,*
$$\langle f, g, h\rangle j = \langle f, g, hj\rangle$$
- *Given $fgh \sim_\epsilon 0$, $hj \sim_{\epsilon'} 0$,*
$$\langle fg, h, j\rangle = \langle f, gh, j\rangle$$
- *Given $fg \sim_\epsilon 0$, $ghj \sim_{\epsilon'} 0$,*
$$\langle f, g, hj\rangle = \langle f, gh, j\rangle$$
- *Given $fg \sim_\epsilon 0$, $gh \sim_{\epsilon'} 0$, $hj \sim_{\epsilon''} 0$,*
$$f\langle g, h, j\rangle = \langle f, g, h\rangle j$$
- *hypothesis*
$$\langle\langle a, b, c\rangle, d, e\rangle + \langle a, \langle b, c, d\rangle, e\rangle + \langle a, b, \langle c, d, e\rangle\rangle$$

**Lemma 4.2.15.** *If $fg = gh = hf = 0$, then*

$$\langle f, g, h\rangle + \langle g, h, f\rangle + \langle h, f, g\rangle$$

**Lemma 4.2.16.**

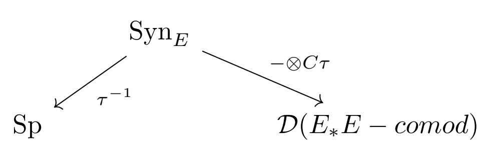
$$\langle (b, c)(d, e), f\rangle = \langle b, c, \langle d, e, f\rangle\rangle$$

INTRODUCE CIRCUMFLEXES

**Lemma 4.2.17.**

**4.2.4. Tensoring down.** Given an element $m \in \pi_* M$ one of the simplest ways to gain information about $m$ is to compute the value of $m$ in $\pi_*(M \otimes R)$ for some ring $R$ which simplifies the situation. In our case of interest two natural choices of $R$ emerge, $R = \mathbb{S}[\tau^{-1}]$ and $R = C\tau$. Using these techniques we will be able to recover and extend results of [???] on the convergence of Massey products in the $E$-based Adams $E_2$-term to Toda brackets. In a very real sense, the entire content of this subsection boils down to the following observation.

**Observation 4.2.18.** There is a diagram of stable psmc and symmetric monoidal left adjoints,




The left arrow sends Toda brackets in the synthetic category to Toda brackets in the usual stable homotopy category. The right arrow sends Toda brackets in the synthetic category to Toda brackets in the category of $E_*E$-comodules (which are sometimes referred to as Massey products).[4]

[4] There are subtleties regarding signs here and they are discussed at length in the section on signs.

As a demonstration of this observation we will provide a proof of the following proposition. We hope that this exercise will clarify the origin of the technical conditions in the statement, and explain how direct examination of synthetic stable stems removes the need for such hypotheses.

**Proposition 4.2.19** ([Mos70])**.** *Let $a_1 \in E_2^{s,t}(X_0, X_1)$, $a_2 \in E_2^{s,t}(X_1, X_2)$ and $a_3 \in E_2^{s,t}(X_2, X_3)$ be permanent cycles such that $a_1 a_2 = 0$ and $a_2 a_3 = 0$. Let $a_1, a_2, a_3$ be realized by maps $\alpha_1, \alpha_2, \alpha_3$ respectively. Suppose also that each of the Adams spectral sequences above converge and [no crossing differentials].*

*Then, the Massey product $\langle a_1, a_2, a_3\rangle$ contains a permanent cycle which is realized by an element of the Toda bracket $\langle \alpha_1, \alpha_2, \alpha_3\rangle$.*

Our proof will consist of rewriting this as a synthetic statement and noting that is true for simple reasons.

- The condition that $a_i$ is a permanent cycle realized by $\alpha_i$ is equivalent to the existence of a synthetic map
$$\widetilde{\alpha_i} : \Sigma^{0,?} \nu X_{i-1} \to \nu X_i$$
such it specialized to $\alpha_i$ and $a_i$ under $\tau^{-1}$ and $- \otimes C\tau$ respectively.
- The vanishing of the product $\alpha_i \alpha_{i+1}$ is equivalent to $\widetilde{\alpha_i}\widetilde{\alpha_{i+1}}$ being $\tau$-power-torsion.
- No crossing differentials condition is equivalent to GROUP being $\tau$-power-torsion free.

Altogether, we obtain maps $\widetilde{\alpha_i}$ and learn that the 3-fold $\langle \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3\rangle$ is defined. The conclusion then follows from the Observation 4.2.18.

EXAMPLES

While one could take this as the go-ahead to prove various "convergence of Massey products to Toda brackets" results in greater generality than previously known, we believe that this misses the essential point. These comparison results are a pale reflection of our ability to lift information from the algebraic category of $C\tau$-modules to the category of synthetic spectra. The point is that serious study of the synthetic category is often <u>easier</u> than the corresponding topological category because of the closer relationship with an algebraic category.

MORE ADVANCED EXAMPLES

Before ending this subsection we give another example where tensoring down along a ring map is useful. This example is supposed to standardize certain manipultations used in the following situation: often one can show that a bracket is "nearly zero" in the sense that its Adams filtration is very high, but the possibility remains that the bracket's value is some non-zero element of the $K(1)$-local homotopy of the sphere. We would like a simple reason that a bracket contains an element which is zero in the $K(1)$-local sphere.

**Example 4.2.20.** Suppose that

- $\alpha, \beta, \gamma \in \pi_* \mathbb{S}$ are nonzero,
- $\langle \alpha, \beta, \gamma\rangle$ is defined,
- $\gamma = 0$ in $L_{K(1)} \mathbb{S}$.

Then, $\langle \alpha, \beta, \gamma\rangle$ contains an element in the kernel of the map to $L_{K(1)} \mathbb{S}$.

Proof. Let $i$ denote the unit map of $L_{K(1)} \mathbb{S}$, then we have an equality
$$\langle \alpha, \beta, \gamma\rangle i = \alpha\langle \beta, \gamma, i\rangle.$$

Let us examine the bracket on the right, it has indeterminacy $(\pi_? \mathbb{S})i + \beta(\pi_? L_{K(1)}\mathbb{S})$. Since $\gamma$ is nonzero in $\pi_*$, but maps to zero in the $K(1)$-local sphere we know that $|\gamma| \geq 8$. This implies the map $i$ is surjective on homotopy groups in the relevant degree so we can always pick a nullhomotopy of $\beta\gamma$ which makes the bracket zero.

□

**4.2.5. Moss' theorem on Massey products.** Previously, we discussed how tensoring down to $C\tau$ can often quickly determine much of the information about a bracket. In many situations this is sufficient, but in others we run into the problem that the indeterminacy becomes too large after tensoring with $C\tau$, as indicated in the following example.

**Example 4.2.21.** Consider the bracket $\langle \sigma^2, 2, \eta \rangle$. Classically, this bracket is well known to have value $\eta_4$ which is detected by $h_1 h_4$ in the Adams spectral sequence. This bracket takes values in $\pi_{16,18}$ therefore we would expect it maps to $h_1 h_4$. Furthermore, consultation with the charts in [location] indicate that there is no indeterminacy. However, on tensoring down to $C\tau$ we obtain $\langle h_3^2, 0, h_1 \rangle$ which isn't useful.

In this section we will explore a technique which makes use of the fact that we started with a spherically defined bracket in order to cut down the indeterminacy. This technique is a synthetic refinement and extension of theorem [which] from Moss [cite]. For a precursor to the material the reader is encouraged to consult [example ?]. We begin with a simple lemma which lies at the heart of the matter.

This seems to belong in the tau bockstein section

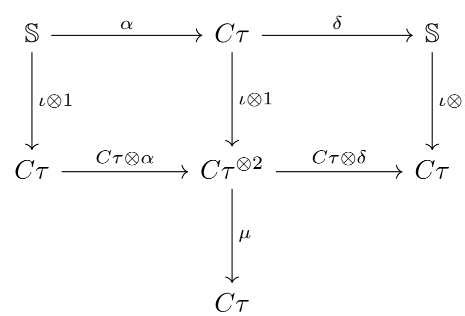
**Lemma 4.2.22.** *Given a map $\alpha : \mathbb{S} \to C\tau$ consider the $C\tau$-linear map*

$$C\tau \otimes \alpha : C\tau \to C\tau \otimes C\tau$$

*In [location] we fixed a choice of equivalence (of $C\tau$-module) between the target of this map and $C\tau \oplus \Sigma^{1,-1} C\tau$. Under this equivalence*

$$(C\tau \otimes \alpha) = (\alpha, \alpha\delta_{1,1}).$$

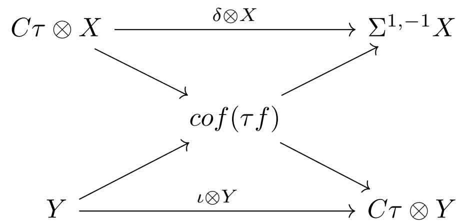
Proof. WLOG since $C\tau$-linear maps out of $C\tau$ are the same as maps out of $\mathbb{S}$ it suffices to precompose with $\iota \otimes 1$ and determine the map. In [location], we fixed two maps out of $C\tau^{\otimes 2}$ the first of these was the multiplication map while the second was projection onto the top cell (tensored up to $C\tau$). The lemma follows from considering the following diagram.



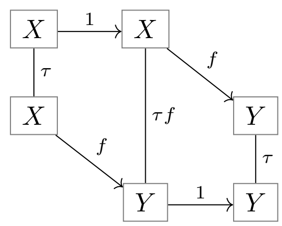

□

Before we can extend this lemma to the cases of interest we will need to fix a choice of splitting as in [location] for more general situations.

**Convention 4.2.23.** Given a map $f : X \to Y$ we fix the following for the splitting of $C\tau \otimes cof(\tau f)$. Using the fact that $\tau f = f\tau$ we construct the diagram indicated below:




The maps along the downward-going diagonal are pictured in the following cell diagram:




Upon tensoring up to $C\tau$ and post-composing with $\mu$ on the bottom right and pre-composing with the dual of $\mu$ on the top left we obtain the desired splittings.

**Lemma 4.2.24.** *Given two maps $f : X \to Y$ and $g : Y \to Z$ and a nullhomotopy of $\tau fg$ we obtain maps $\hat{f} : X \to fib(\tau g)$ and $\hat{g} : cof(\tau f) \to Z$ by extension. After tensoring these maps with $C\tau$ and using the splitting of convention 4.2.23 we obtain equalities*

$$(C\tau \otimes \hat{f}) = (f, h) \quad \text{and} \quad (C\tau \otimes \hat{g}) = \begin{pmatrix} h \\ g \end{pmatrix}$$

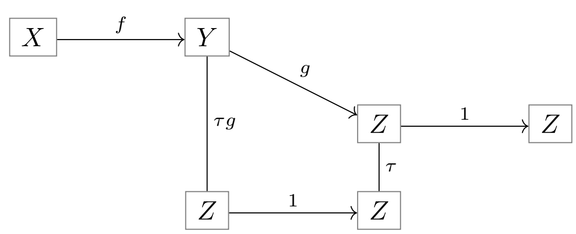
*For some map $h$ such that $h\delta = fg$.*

- *Modifying the nullhomotopy by $k$ has the effect of modifying $h$ by $k\iota$.*
- *If $fg = 0$ and the nullhomtopy of $\tau fg$ is given by multiplying a nullhomotopy of $fg$ by $\tau$ then $h = 0$.*

move to CtauN in this lemma

Proof. We will prove only the first equality, the second being dual to the first. WLOG since $C\tau$-linear maps out of $C\tau \otimes X$ are the same as maps out of $X$ it suffices to precompose with $\iota \otimes 1_X$ and determine the map. In convention 4.2.23, we fixed two maps in and out of $C\tau \otimes cof(\tau g)$. The map out to the copy of $C\tau \otimes Y$ is given by the map out to the top cell followed by $\iota$ so we obtain the first part by the definition of extension. For the second part consider the following composite and the cell diagram representing it below:

$$X \xrightarrow{\hat{f}} cof(\tau g) \longrightarrow C\tau \otimes Z \xrightarrow{\delta} Z$$




□

**Proposition 4.2.25.** *Given maps $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ and nullhomotopies of $\tau fg$ and $\tau gh$ we can define the bracket $k := \langle f, g, h \rangle$. Then, there exist $C\tau$-linear maps*

*$a : C\tau \otimes X \to C\tau \otimes Z$ and $b : C\tau \otimes Y \to C\tau \otimes W$ such that $\iota a\delta = fg$, $\iota b\delta = gh$ and $k\iota = fb + ah$.*

**Remark 4.2.26.** In the situation where $X, Y, Z, W$ are all bigraded shifts of objects in the image of $\nu$ we may give a classical interpretation of the proposition. In this guise this is [Moss' thm]. Using the identification of the $\tau$-bockstein spectral sequence with the decalage of the $\nu E$-Adams spectral sequence we may identify $a$ and $b$ as element of the $E_2$-term such that $d_r(a) = fg$ and $d_r(b) = gh$ respectively. Note that the restriction on crossing differentials present in Moss' formulation of this result is a way to ensure that $\tau fg = 0$ iff it is zero after inverting $\tau$.

PROOF. Unwinding [defintion] we are evaluating the the composite

$$X \xrightarrow{\hat{f}} cof(\tau g) \xrightarrow{\hat{h}} W$$

After tensoring with $C\tau$ the middle term splits and by lemma 4.2.24 this becomes

$$C\tau \otimes X \xrightarrow{(f,a)} (C\tau \otimes Y) \oplus (C\tau \otimes Z) \xrightarrow{\begin{pmatrix} b \\ h \end{pmatrix}} C\tau \otimes W$$

with composite given by $fb + ah$. □

**Example 4.2.27.** We now examine the example from the start of this section $\langle \sigma^2, \widetilde{2}\tau, \eta \rangle$. As before we can read off from [examples chapter chart] that this bracket is defined. For ease of evaluation we can parenthesize the bracket: $\langle \sigma^2, \tau(\widetilde{2}, \eta) \rangle$. Now applying proposition 4.2.25 we obtain that $a = h_4$ and $b = 0$ so this bracket maps to $h_1 h_4$ in $C\tau$ as desired.

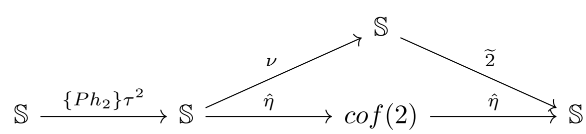
**Example 4.2.28.** Using the differential $d_3(h_0^2 h_4) = h_2 P h_2$ we obtain

$$h_0^5 h_4 \in \langle \{Ph_2\}, \tau^2(\nu, \widetilde{8}) \rangle.$$

Let's expand this bracket a bit

$$\left\langle \{Ph_2\}, \tau^2(\nu, \eta), \begin{pmatrix} \widetilde{4} \\ \eta \end{pmatrix} \right\rangle$$

Let's expand this bracket again,




Shuffling $\tau^2$ inward we can apply proposition 4.2.25 to obtain that this bracket is $h_0^3 h_4$. finish this example

FIND EXAMPLE WITH TWO TERMS APPEARING

Next we turn to the analog of Moss' theorem for 4-folds. At this point we make genuine gains over previous formulations since it may not be possible to form such a 4-fold meaninfully on any particular page of the Adams spectral sequence.

**Proposition 4.2.29.** *Given the following data*

- *Let $f_1, f_2, f_3, f_4$ denote composable maps between synthetic spectra $X_0, X_1, X_2, X_3, X_4$.*
- *Let $\epsilon_1, \epsilon_2, \epsilon_3$ denote nullhomotopies of $\tau^{n_1} f_1 f_2$, $\tau^{n_2} f_2 f_3$ and $\tau^{n_3} f_3 f_4$ respectively.*
- *Let $m = \max(n_1, n_2 - n_3)$, then we may form 3-folds*

$$\langle f_1, \tau^m f_2, \tau^{n_3} f_3 \rangle \quad \text{and} \quad \langle \tau^m f_2, \tau^{n_3} f_3, f_4 \rangle$$

*using the $\epsilon_i$.*

- *Let* $\rho_1$ *and* $\rho_2$ *denote nullhomotopies of* $\tau^{s_1}\langle f_1, \tau^m f_2, \tau^{n_3} f_3\rangle$ *and* $\tau^{s_2}\langle \tau^m f_2, \tau^{n_3} f_3, f_4\rangle$ *respectively.*
- *Finally, let* $s = \max(s_1, s_2)$*,* $p_1 = m$ *and* $p_2 = n_3 + s$*.*

*we can form the synthetic 4-fold bracket* $q := \langle f_1, \tau^{p_1} f_2, \tau^{p_2} f_3, f_4\rangle$*. As earlier, for each of the nonzero* $n_i$ *we can form maps using the given nullhomotopies,*

- $a_1 : X_0 \to C\tau^{n_1} \otimes X_2$*, with* $a_1\delta = f_1 f_2$*.*
- $a_2 : X_1 \to C\tau^{n_2} \otimes X_3$*, with* $a_2\delta = f_2 f_3$*.*
- $a_3 : X_2 \to C\tau^{n_3} \otimes X_4$*, with* $a_3\delta = f_3 f_4$*.*

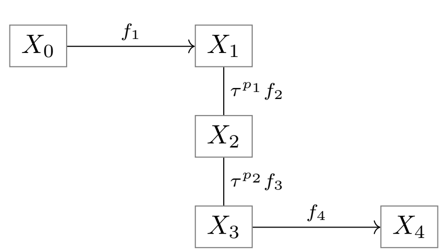
*Furthermore, for each nonzero* $s_i$ *we can form maps using the given nullhomotopies,*

- $b_1 : X_0 \to C\tau^{p_1} \otimes X_3$*, with* $b_1\delta = \langle f_1, \tau^m f_2, \tau^{n_3} f_3\rangle$*.*
- $b_2 : X_1 \to C\tau^{p_2} \otimes X_4$*, with* $b_2\delta = \langle \tau^m f_2, \tau^{n_3} f_3, f_4\rangle$*.*

*Finally, we have the following simplification of the bracket after smashing with* $C\tau$*:*

- *If* $p_1, p_2 \neq 0$*,* $n_2 < p_1 + p_2$ *and* $s \neq 0$*, then*
$$q = \tau^{s-s_2} f_1 b_2 + \tau^{s-s_1} b_1 f_4.$$
- *If* $p_1, p_2 \neq 0$*,* $n_2 < p_1 + p_2$ *and* $s = 0$*, then*
$$q = \tau^{s-s_2} f_1 b_2 + \tau^{s-s_1} b_1 f_4.$$
- *If* $p_1, p_2 \neq 0$ *and* $n_2 = p_1 + p_2$*, then*
$$q = \tau^{\max(n_1+n_3, n_2)+s} a_1 a_3 + \langle f_1, a_1, f_4\rangle.$$
- *If* $p_1 = 0$ *and* $n_2 < p_1 + p_2$*, then*
$$q = \langle f_1, f_2, a_3\rangle + b_1 f_4.$$
- *If* $p_1 = 0$ *and* $n_2 \geq p_1 + p_2$*, then*
$$q = \left\langle f_1, (f_2, a_2), \begin{pmatrix} a_3 \\ f_4 \end{pmatrix} \right\rangle.$$
- *If* $p_2 = 0$ *and* $n_2 < p_1 + p_2$*, then*
- *If* $p_2 = 0$ *and* $n_2 \geq p_1 + p_2$*, then*
- *If* $p_1, p_2 = 0$*, then no simplification is possible.*

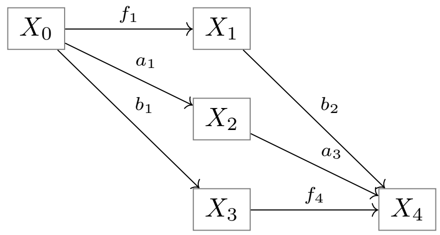
Proof. Our goal is to evaluate the bracket $\langle f_1, \tau^{p_1} f_2, \tau^{p_2} f_3, f_4\rangle$ after tensoring with $C\tau$. We begin by noting that the choice of nullhomotopy of $\tau^{p_1} f_2 \tau^{p_2} f_3$ gives us a specific 3-cell complex which we will call $Y$. The chosen nullhomotopy of $\langle f_1, \tau^{p_1} f_2, \tau^{p_2} f_3\rangle$ provides a map $X_0 \to Y$. The chosen nullhomotopy of $\langle \tau^{p_1} f_2, \tau^{p_2} f_3, f_4\rangle$ provides a map $Y \to X_4$. The value of the 4-fold we wish to evaluate is given by the composite of these maps. We display a cell diagram illustration of this below.



Let $q$ denote the value of the bracket. We break into six cases based on what happens when we tensor $Y$ with $C\tau$.

**Case 1** $p_1, p_2 \neq 0$ and $n_2 < p_1 + p_2$.

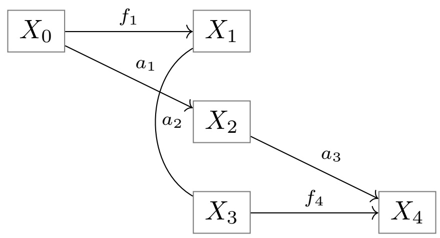
In this case we obtain a cell diagram of the following form after tensoring with $C\tau$,




We can read off that $q = f_1 b_2 + a_1 a_3 + b_1 f_4$.

**Case 2** $p_1, p_2 \neq 0$ and $n_2 \geq p_1 + p_2$.

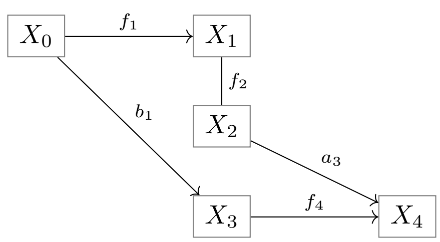
In this case we obtain a cell diagram of the following form after tensoring with $C\tau$,




We can read off that $q = a_1 a_3 + \langle f_1, a_1, f_4 \rangle$.

**Case 3** $p_1 = 0$ and $n_2 < p_2$.

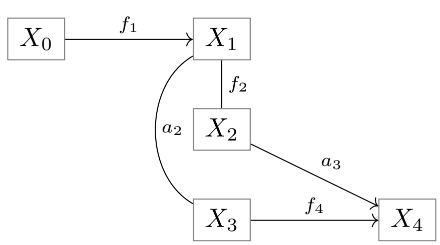
In this case we obtain a cell diagram of the following form after tensoring with $C\tau$,




We can read off that $q = \langle f_1, f_2, a_3 \rangle + b_1 f_4$.

**Case 4** $p_1 = 0$ and $n_2 \geq p_2$.

In this case we obtain a cell diagram of the following form after tensoring with $C\tau$,




We can read off that $q = \left\langle f_1, (f_2, a_2), \begin{pmatrix} a_3 \\ f_4 \end{pmatrix} \right\rangle$.

**Case 5** $p_2 = 0$ and $n_2 < p_1$. This case is the same as case 3 up to symmetry.
**Case 6** $p_2 = 0$ and $n_2 \geq p_1$. This case is the same as case 4 up to symmetry. □

The statement of proposition 4.2.29 is significantly more difficult to follow than the proof. For this reason we have decided not to continue exploring all variations on Moss' theorem provable at this moment. Instead, we suggest that with the tools provided readers may analyze brackets of arbitrary complexity directly.

EXAMPLES

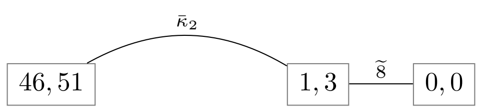
**Remark 4.2.30.** Given a Toda bracket defined in the usual category of spectra (for concreteness we can assume this is a 3 or 4-fold bracket), one might try to produce another bracket in the category of synthetic spectra which reduces to the given one upon inverting $\tau$. A measure of the quality of such a lift is given by how many times it is "divided by $\tau$" as compared with the lift produced by the functor $\nu$.

Upon picking a lift (which is viewed as optimal) we refer to the value of this bracket after tensoring down to $C\tau$ as the "primary value". Note that this notion of primary value often differs from simply picking classes on an $E_2$-page representing the maps and evaluating a Massey product.

From this viewpoint, the material in this subsection was about algebraically determining the primary value of various brackets. In general, we have found when the prmary value of a bracket is zero, determining the secondary value can be quite difficult. It seems likely that if any secondary value can be computed in terms of a longer sequence of questions about primary values, then there could exist an effective algorithm for determining the stable homoopty groups of spheres. We leave this possibility as a question.

**4.2.6. Example: the operator M.** It is an insight of Dan isaksen to recognize the importance of the operator $M(-)$ in understanding the homotopy groups of spheres [is this the right citation?]. The Massey product analog of $M(-)$ is very often nonzero on the $\mathbb{F}_2$-Adams $E_2$ page and many of the longest known differentials hit $M(-)$ applied to well-known classes. One of the salient features of $M(x)$ is that it is often equal to the product $\theta_{4.5}x$. In this example we will explore an $\mathbb{F}_2$-synthetic analog of $M$ and in certain situations relate it to multiplication with $\theta_{4.5}$.

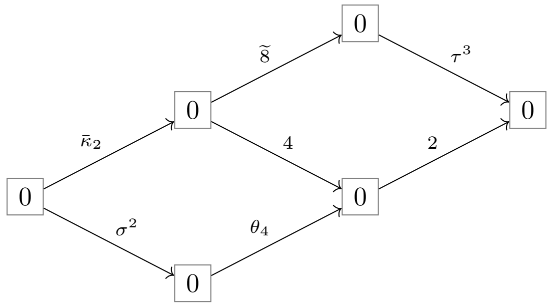
**Definition 4.2.31.** Let $M(-)$ denote the $\mathbb{F}_2$-synthetic bracket based on the following complex,

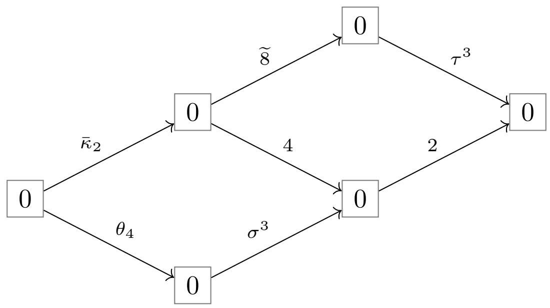



Inverting $\tau$ recovers the definition of $M(-)$ from [location]. Modding out by $\tau$ recovers the definition of $M$ as a Massey product. Based on this we hope no confusion will result from our choice of notation.

In order to understand the relation between $M(-)$ and $\theta_{4.5}$ we will produce a bracket whose value is the $\mathbb{F}_2$-synthetic element $\theta_{4.5} \in \pi_{45,48}$.

**Lemma 4.2.32.** *The following pair of $\mathbb{F}_2$-synthetic 3-fold are defined and both have the property that their value maps to $h_4^3$ in $C\tau$.*




*Let $\alpha_1$ and $\alpha_2$ denote the values of the first and second brackets respectively. The indeterminacy of the first bracket is [indet]. The indeterminacy of the second bracket is [indet]. The two are related by $\alpha_1 - \alpha_2 \in \langle \theta_4, 2, \sigma^2 \rangle$. Furthermore, they are related to the definiton of $\theta_{4.5}$ from [cite] by [it's the first one].*

It is notable that this 3-fold bracket for $\theta_{4.5}$ uses the element $\tau^3$ in a spot where it contributes to the indeterminacy. Classically, this bracket would have indeterminacy $\pi_{45} \cdot 1$. We find this a nice demonstration of the principle that synthetic indeterminacy is often smaller than classical indeterminacy.

PROOF. □ write proof.

**Lemma 4.2.33.** *Suppose $\alpha \in \pi_{**}$ has the property that $\bar{\kappa}_2 \alpha = \widetilde{8}\alpha = 0$,*

- *if $\sigma^2 \alpha = 0$, then $\theta_{4.5}\alpha = \tau^3(\mathbb{D}M)(\alpha) + 2\Theta_{3,4}(\alpha)$*
- *if $\theta_4 \alpha = 0$, then $\theta_{4.5}\alpha = \tau^3(\mathbb{D}M)(\alpha) + 2\Theta_{3,4}(\alpha)$*

Using the equation $(\mathbb{D}M)(-) + M(-) + Comm(\widetilde{8}, \bar{\kappa}_2)(-) = 0$ we can use this lemma to relate $M(-)$ and multiplication by $\theta_{4.5}$ up to the given error terms.

PROOF. For the first bullet point shuffle the first bracket from lemma 4.2.32. For the second bullet point shuffle the second bracket from lemma 4.2.32. □

With the set-up out of the way we can now go about relating $M$ to multiplication by $\theta_{4.5}$ in practice. We will spend the rest of this section examing the cases of $M(\eta), M(\nu), \{MP\}, M(\kappa)$ and $M(\bar{\kappa})$. Not all of these can be handles by lemma 4.2.33 directly

**Example 4.2.34.** Consider $\nu^2$, by consulting a chart one can conclude that $\widetilde{2}\nu^2$, $\nu\sigma$ and $\nu^2\bar{\kappa}_2$ are zero. Then, by lemma 4.2.33 we know that

$$\tau^3 M(\nu^2) = \tau^3(\mathbb{D}M)(\nu^2) - \tau^3 Comm(\widetilde{8}, \bar{\kappa}_2)(\nu^2) + 2\Theta_{3,4}(\nu^2)$$

Now we run through each of the terms in the above. finish this

Using this piece of information we can use it to understand various other extensions. Since $\nu\theta_{4.5}$ must now be nonzero

**Example 4.2.35.** Consider $\kappa$, Consider $\bar{\kappa}$,

**Example 4.2.36.** Suppose that $h_3h_5$ lifted to a class $\alpha$ in $C\tau^4$. Then, we would have $\sigma\alpha = \theta_{4.5}+?\tau\{h_5d_0\}+?\eta\bar{\kappa}_2$. Multiplying this by $\nu$ we get

$$0 = \nu\theta_{4.5}+?\tau\nu\{h_5d_0\} = \tau^3M(\nu)+?\tau\nu\{h_5d_0\} \neq 0.$$

The contradiction implies that $\delta_4(h_3h_5) \neq 0$. For sparsity reasons we may conclude that $\delta_4(h_3h_5) = h_0x$.

## 4.3. The Liebniz rule

One of the most useful techniques for evaluating Adams differentials is the Liebniz rule. In the $\mathbb{F}_2$-Adams spectral sequence this alone reduces the number of $d_2$'s one needs to compute through dimension 50 from 344 to 66.[5] In it's most basic form, the Liebniz rule says that if $a$ is a permanent cycle then $d_r(ab) = ad_r(b)$. In this section we demonstrate that using synthetic spectra much stronger forms of "linearity" of differentials can be proved.

**4.3.1. The Liebniz rule for the sphere.** We begin by exploring the Liebniz rule for $\tau$-bockstein differentials in the bigraded homotopy of the sphere. The theoretical aspects of this subsection already appeared in [location] and particularly **??** where the explicit formula

$$\delta_{2N,N}(ab) = a\delta_{2N,N}(b) + \delta_{2N,N}(a)b$$

was proved for $a, b \in \pi_{**}C\tau^N$. Here we devote ourselves to the more practical concern of demonstrating the utility of this and similar formulas. We begin with the following which we regard as a more creative use of the Liebniz rule.

**Example 4.3.1.** From [example in moss section] we know that the bracket $\langle \eta, 2, \sigma^2 \rangle$ is defined and maps to $h_1h_4$ in $C\tau$. In [location] we will show that $\delta(h_0h_4) = 2\kappa$. Using the Liebniz rule we have the following equalities in $C\tau^2$,

$$\begin{aligned} \tau\eta^3\kappa = \widetilde{2}\nu 2\kappa &= \widetilde{2}\nu\delta_2([h_0h_4]) + \delta(\widetilde{2}\nu)[h_0h_4] = \delta(\widetilde{2}\nu[h_0h_4]) \\ &= \delta(\eta^2[h_1h_4]) = \eta^2\delta([h_1h_4]) + \delta(\eta^2)[h_1h_4] = 0 \end{aligned}$$

For sparsity reasons we then learn that $\delta(h_1e_0) = \eta^3\kappa$. Applying the Liebniz rule one more time we learn that $\delta(e_0) = \eta^2\kappa$.

There's a subtle annoyance here...

So far when we use the Liebniz rule we've been in the convenient situation where our pair of elements $x$ and $y$ both live in $C\tau^N$. We next explain what can be done when $x$ and $y$ live in different places.

**Observation 4.3.2.** Suppose we are given $x \in \pi_{**}C\tau^n$ and $y \in \pi_{**}C\tau^m$ with $n \geq m$. The natural way to apply the Liebniz rule would be to first project $x$ down to $C\tau^m$ using $r$ and then consider $\delta$ applied to the product $r(x)y$. Carrying out this computation using the above proposition we learn that

$$\begin{aligned} \delta_{2m,m}(r_{n,m}(x)y) &= \delta_{2m,m}(r_{n,m}(x))y + (-1)^{|x|}r_{n,m}(x)\delta_{2m,m}(y) \\ &= \tau^{n-m}\delta_{n+m,n}(x)y + (-1)^{|x|}r_{n,m}(x)\delta_{2m,m}(y) \end{aligned}$$

[5] These numbers refer to the dimension of the appropriate $\mathbb{F}_2$-vector space and we've excluded $\pi_0$ in order to get a finite answer.

Note that the output lives in $C\tau^m$ which may not be particularly useful if $n \gg m$. A finer result can be obtained if we instead use the dual of $r$ to push $y$ into $C\tau^n$ and take the product there,

$$\begin{aligned}\delta_{2n,n}(x(\mathbb{D}r_{n,m})(y)) &= \delta_{2n,n}(x)(\mathbb{D}r_{n,m})(y) + (-1)^{|x|}x\delta_{2n,n}(\mathbb{D}r_{n,m})(y) \\ &= \delta_{2n,n}(x)(\mathbb{D}r_{n,m})(y) + (-1)^{|x|}x\delta_{n+m,m}(y)\end{aligned}$$

This observation doesn't cover the case when one $n = \infty$, however that case is in fact even simpler.

**Observation 4.3.3.** Since $\delta : C\tau \to \mathbb{S}$ is a map of synthetic spectra it is naturally linear for the action of the synthetic sphere. From this we learn the following: Suppose that $b \in \pi_{**}C\tau$ and $\alpha \in \pi_{**}$. Then, $\delta(\alpha b) = \alpha\delta(b)$.

The power of the latter observation comes from the fact that no mention of "page", "length" or "$C\tau^N$" was necessary.

**Example 4.3.4.** Picking up where example 4.3.1 left off, we know that $\delta(e_0) = \eta^2\kappa$. Now in order to compute the $d_4$ differential on $e_0 g$ we can proceed as follows:

$$\delta(ge_0) = \delta(\bar{\kappa}e_0) = \bar{\kappa}\delta(e_0) = \bar{\kappa}\eta^2\kappa = \tau^2\{Pd_0\}\kappa$$

where we've used that $g$ is a permanent cycle detecting $\bar{\kappa}$ [cite], and the relation $\tau^2\{Pd_0\} = \eta^2\bar{\kappa}$ [cite].

**Example 4.3.5.** Consider the differential on $h_2h_5$,

$$\delta(h_2h_5) = \delta(\nu h_5) = \nu\delta(h_5) = \nu\widetilde{2}\theta_4.$$

In order to compute this differential it will now suffice to understand $\nu\theta_4$.

Consider the following Massey product for $c_2$,

$$c_2 = \langle h_4^2, h_2, h_3\rangle$$

If $\nu\theta_4$ was zero, then we would be able to upgrade this Massey product to a syntethic Toda bracket. On the other hand, from power operations we know that $d_2(c_2) = h_0f_1 = h_3p$ (this appears as [exm] in [location]). This is enough to conclude that $\nu\theta_4$ is nonzero, but we can actually go furhter. Since $c_2$ does not lift to the $C\tau^2$ we know that this Massey product cannot be lifted to a synthetic toda bracket in $C\tau^2$. Thus, we learn $\nu\theta_4 \neq 0$ in $C\tau^2$. From this we may conclude that there exists a class $p \in \pi_{33,37}$ which maps to the usual $p$ in $C\tau$ and such that $\tau p = \nu\theta_4$. Putting everythng together we can now conclude that

$$\delta(h_2h_5) = \nu\widetilde{2}\theta_4 = \tau\widetilde{2}p$$

**Example 4.3.6.** Consider the differential on $h_3h_5$ which was calculated in example 4.2.36. From it we obtain,

$$\tau^2[h_0x] = \delta_4(h_3h_5) = \delta(\sigma h_5) = \sigma\widetilde{2}\theta_4.$$

which implies that $\tau^2\{x\} = \sigma\theta_4$.

**4.3.2. The Liebniz rule in general.** Having studied the specific case of the sphere we now turn to understanding the Liebniz rule in general. We will find that once things are set up appropriately the sphere was in fact the universal example.

Although we have discussed the way in which the $\tau$-bockstein

We will also need a slight enhancement of this lemma,

**Lemma 4.3.7.** *Given $C\tau^N$-linear map*

$$\alpha : C\tau^N \to C\tau^N$$

*there is an equivalence*

$$(\alpha \otimes C\tau^N) = \begin{pmatrix} \alpha & \alpha\delta_{2N,N} \\ 0 & \alpha \end{pmatrix}$$

PROOF. The splitting of $C\tau^N \otimes C\tau^N$ is induced by the pair of maps $(m, \delta_{2N,N} \otimes C\tau^N)$ where $m$ is the multiplication map.

One of the diagonal entries should be clear. The other follows by duality. The top left entry essentially comes from our choice of splitting. The upper triangularity follows from the $C\tau^N$-linearity of $\alpha$. □

**Theorem 4.3.8.** *On the one hand*

$$\langle a, b, c\rangle \otimes C\tau^N = \begin{pmatrix} \langle a, b, c\rangle & \langle a, b, c\rangle\delta_{2N,N} \\ 0 & \langle a, b, c\rangle \end{pmatrix}$$

*while on the other*

$$\langle a, b, c\rangle \otimes C\tau^N = \left\langle \begin{pmatrix} a & a\delta_{2N,N} \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & b\delta_{2N,N} \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & c\delta_{2N,N} \\ 0 & c \end{pmatrix} \right\rangle$$

PROOF. The functor

$$(- \otimes C\tau^N) : \mathrm{Mod}_{C\tau^N} \to \mathrm{Mod}_{C\tau^N}$$

is exact and therefore sends Toda brackets to Toda brackets. Then, the theorem follows by applying Lemma 4.3.7 □

**Lemma 4.3.9.** *If $R$ is an $\mathbb{A}_2$ ring, then*

$$\delta_{2N,N}(m(x, y)) = m(x, \delta_{2N,N}(y)) + m(\delta_{2N,N}(x), y)$$

PROOF. The map $m(x, y)$ can be expanded as

$$C\tau^N \otimes_{C\tau^N} C\tau^N \xrightarrow{1\otimes y} C\tau^N \otimes_{C\tau^N} R \xrightarrow{x\otimes 1} R \otimes_{C\tau^N} R \xrightarrow{m} R.$$

Upon applying [???] this expansion becomes the equality

$$\begin{pmatrix} m(x, y) & \delta(m(x, y)) \\ 0 & m(x, y) \end{pmatrix} = \begin{pmatrix} 1 \otimes y & \delta(1 \otimes y) \\ 0 & 1 \otimes y \end{pmatrix} \begin{pmatrix} x \otimes 1 & \delta(x \otimes 1) \\ 0 & x \otimes 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

□

### 4.3.3. Example: the $K(1)$-local sphere.

CHAPTER 5

# Powers

CHAPTER 6

# A closer look at $\mathbb{S}$, $\mathbb{F}_p$ and BP

The origin of this book lies in the study of the Adams and Adams–Novikov spectral sequences computing the homotopy groups of the sphere. In this chapter we return to these roots more explicitly. We recast a substantial body of previous work in terms of the two examples $\mathrm{Syn}_{\mathrm{BP}}$ and $\mathrm{Syn}_{\mathbb{F}_p}$ and examine these categories closely from several perspectives.

In the first three sections we study the role played by "slopes" in the structure of these categories. This begins with a general study of slopes in Section 6.1. Then, in Sections 6.3 and 6.4 we make a computational study of the first several slopes. We close with Sections 6.5 and 6.6 where we apply the techniques developed throughout this work to recompute the stable homotopy groups of spheres through degree 50 via the Adams and Adams–Novikov spectral sequence respectively.

**A brief history.**

Unlike the previous chapters, which may each be viewed as the development from first principles of a single core idea, the sections of this chapter are more disparate and it is only with sufficient historical context and some imagination on the part of the reader that they can be woven into a convincing whole.

Following Adem's work, the first non-existence result for maps of Hopf invariant one was Toda's proof that there does not exist an element of Hopf invariant one in $\pi_{15}$. This result was obtained as part of Toda's computations of the homotopy groups of spheres (both stable and unstable) [Tod55] [Tod62].

MORE HERE

For the purpose of this introduction we note that there are essentially six basic approaches to studying the homotopy groups of spheres and give an overview of each approach to provide context for the somewhat disparate sections that follow.

| | | |
|---|---|---|
| Adams sseq<br>below topological degree $n$ | Adams sseq<br>below filtration $s$ | Adams sseq<br>above a line of slope $m$ |
| Adams–Novikov sseq<br>below topological degree $n$ | Adams–Novikov sseq<br>below filtration $s$ | Adams–Novikov sseq<br>above a line of slope $m$ |

The first column is almost self-explanatory and involves a finite amount of algebra in order to compute the $E_2$-page (often undertaken with computer assistance) followed by the topological task of computing differentials and hidden extensions. Tangora's computation of the $E_2$-term through 70 via the May sseq remains essentially unsurpassed (by a human) [Tan70]. Machine calculations have progressed significantly farther, for the state of the art see [many cites]. The computation of differentials in the Adams spectral sequence was undertaken by Mark Mahowald and others with great vigor throughout the late 20$^{\text{th}}$ century [cites]. For the state of the art see [IWX20] which is essentially complete through the 80 stem. In Section 6.5 we will use the techniques developed in this book to recompute the differentials and hidden extensions through degree 50. Our reason for stopping at 50 is that beyond this point the number of classes on the $E_2$ page becomes substantial.

The process of studying the Adams–Novikov sseq one line at a time is essentially equivalent to the chromatic approach to studying stable homotopy theory. This has its origin in [**Ada66a**] where Adams studied the image of $J$ through the $d$ and $e$ invariant. In [**Rav86**, location], Ravenel identifies the $d$ and $e$ invariant as capturing the 0 and 1 line of the Adams–Novikov sseq. The study of the 2-line in the sphere was carried out in [**MRW77**] while the Japanese school constructed a variery of different spectra equipped with non-nilpotent self-maps. Based on this evidence Ravenel put forward a bold collection of conjectures which essentially assert that the theory of heights for formal groups plays a dominant role in the large scale behavoir of the category of spectra. These conjectures were proved in short order by Devinatz, Hopkins and Smith whose nilpotence, periodicity and chromatic convergence theorems essentially verified Ravenel's vision. Since this subject is so well-studied and due to space limitations we will not presently investigate what leverage synthetic methods provide in studying finite-height phenomena. We suggest the interested reader consult the green and orange books for an introduction to this subject [**Rav86**] [**Rav92**].

Surprisingly little is known about the Adams spectral sequence in low filtrations and what is known was hard-won. At the prime 2 Adams' solution of the Hopf invariant one problem [**Ada60**] can be rephrased as showing that the element $h_j$ on the $E_2$ page support $d_2$ differentials for $j \geq 4$. Thus, the 1-line of the Adams sseq has only 4 nontrivial elements 2, $\eta$, $\nu$ and $\sigma$ [1]. In [**Mah77**], Mahowald used Brown–Gitler spectra to construct an infinite family of elements $\eta_j$ which are detected by $h_1 h_j$ on the 2-line of the Adams sseq. Mahowald's construction of the $\eta_j$ family provided a counterexample to the so-called doomsday conjecture that every line of the Adams spectral sequence has only finitely many elements. More recently Hill, Hopkins and Ravenel have shown that the elements $h_j^2$ supports non-trivial differentials for $j \geq 7$ [**HHR16**] [2].

The algebraic task of computing the $E_2$-page one line at a time has been taken up by Lin and his students. For example, in [**Lin08**] and [**Che11**] they determine up to the 5-line at $p = 2$. Examining these output of these computations one finds that the structure of the $E_2$-page of the Adams sseq for the sphere is dominated (in low filtrations) by the existence of the power operation $Sq^0$, which corresponds to the Frobenius on the stack of additive formal groups in characteristic $p$[3] (for example, $Sq^0(h_j) = h_{j+1}$). If we refer to the collection of elements $(Sq^0)^n(x)$ as the $Sq^0$ family generated by $x$, then, we have the following replacement for the original doomsday conjecture:

**Conjecture 6.0.1** (Minami's new doomsday conjecture)**.** *Only finitely many elements in any $Sq^0$ family detect non-trivial elements of the homotopy groups of spheres.*

This conjecture is striking for several reasons. First, the author knows of no reason, heuristic or otherwise, to believe that this conjecture should be true. Second, while much research in the past several decades has centered on exploring the category of spectra by progressing up the chromatic tower, this conjecture is *anachromatic* in the sense that it cuts accross heights in a way that need not be controlled by any particular height.

[1] As discussed in [location] these are the only elements of the 2-local stable stems of depth infinity and this implies that every other element of the stable stems is "decomposable in terms of Hopf invariant one elements". We again warn the reader that this notion of decomposable it not particularly workable in practice.

[2] Note that no bound on the lengths of the differentials is provided.

[3] A more complete discussion of this operation appears in [location].

SLOPES IN ADAMS SSEQ
SLOPES IN AN SSEQ

## 6.1. Slopes and vanishing lines

In this section we will explore a collection of distinguished $\otimes$-ideals in (compact) synthetic spectra provided by "vanishing lines" in the bigraded homotopy groups. These $\otimes$-ideals were first constructed in [HPS99] where a connection with (and interpretation of) the $\otimes$-ideals in Sp is provided. These same ideas also lie at the heart of the Hopkins–Smith construction of $v_n$-periodicity maps. The basic definition is relatively simple and its interpretation in terms of an $E$-Adams sseq is also clear.

**Definition 6.1.1.** We will say that a synthetic spectrum $X$ admits a *vanishing line* of slope $m$ and intercept $c$ if $\pi_{k,s}(X) = 0$ for $s > mk + c$. We will say that a synthetic spectrum $Y$ has a *finite-page vanishing line* of slope $\mu$, intercept $c$ and torsion level $r$ if every element of $\pi_{k,s}(Y)$ with $s > \mu k + c$ is $\tau^r$-torsion.

In order to illustrate the definition we provide three examples and supporting charts.

**Example 6.1.2.** For any $m$ the cofiber of $\tau$ admits a finite-page vanishing line of slope $m$, intercept $-\infty$ and torsion level 1.

CHART HERE

**Example 6.1.3.** The unit in $\mathrm{Syn}_{\mathrm{BP}}$ admits a vanishing line of slope 1 and intercept 0.

CHART HERE

**Example 6.1.4.** In $\mathrm{Syn}_{\mathbb{F}_p}$, the cofiber of $\widetilde{p}$ admits a vanishing line of slope $q^{-1}$ and intercept $q^{-1}$ (for $p = 2$ the intercept is $3/2$ instead).

CHART HERE

In these charts we see that typically a given object will admit many different vanishing lines simultaneously. In order to capture this range of behaviors we make the following collection of definitions.

**Definition 6.1.5.**

- Given an $e \in [0, \infty]$ we will let $\mathrm{Syn}_E^{m=e}$ denote the full subcategory of objects which admit a vanishing line of slope $e$.
- Given an interval $I \subset [0, \infty]$ we will let $\mathrm{Syn}_E^{m=I}$ denote the full subcategory of objects which admit a vanishing line of slope $e$ for every $e \in I$.
- Given an $\epsilon \in [0, \infty]$ we will let $\mathrm{Syn}_E^{\mu=\epsilon}$ denote the full subcategory of objects which admit a finite-page vanishing line of slope $\epsilon$.
- Given an interval $I \in [0, \infty]$ we will let $\mathrm{Syn}_E^{\mu=I}$ denote the full subcategory of objects which admit a finite-page vanishing line of slope $\epsilon$ for every $\epsilon \in I$.

**Remark 6.1.6.** Note that an object is in $\mathrm{Syn}_E^{m=[a,b]}$ if and only it admits a vanishing line of slope $a$ and another vanishing line of slope $b$, i.e. only the endpoints of the interval matter.

**Remark 6.1.7.** In this section we will often restrict our attention to compact objects. Our reason is that infinite sums of bigraded shifts of an object can create "artificial" vanishing lines whose presence and study does not further our understanding of $\mathrm{Syn}_E$.

The first subsection focuses on the elementary properties of vanishing lines which hold for most choices of $E$. The second subsection focuses on the specific cases of BP and $\mathbb{F}_p$. In this subsection the previous algebraic work of Palmieri and Krause provides much of the necessary input [**?**] [**?**]. The third subsection introduces "slope localizations" and uses this framework to sharpen several earlier results. The final subsection examines the the telescope conjecture in this setting. In particular, we are able to show that the $\mathbb{F}_p$-synthetic telescope conjecture is true, though as it turns out this does not have a direct relationship with the telescope conjecture in the category of spectra. We will examine what leverage this results provides on the telescope conjecture more closely in forthcoming work [**?**].

### 6.1.1. Vanishing lines are generic.

In this section we explore the basic properties of vanishing lines. For us this means analyzing how vanishing lines change under (co)limits and tensor products. As a consequence of this analysis we conclude that (finite-page) vanishing lines depend only on relatively weak information about a synthetic spectrum. In fact, it will turn out that vanishing lines on $X$ essentially only depend on $C\tau \otimes X$ (i.e. they are algebraic) while finite-page vanishing lines on $X$ essentially only depend on $X[\tau^{-1}]$ (i.e. they only see classical data). The interplay between these two sides is what leads to much of the richness of this subject.

The material in this subsection was developed jointly with Senger and Hahn as a reinterpretation of the main results of [**HPS99**] in the synthetic language and first appeared in [**BHS19**, Section 11] [4] [5].

**Note 6.1.8.** In this version we will assume that $E$ is either BP or $\mathbb{F}_p$ throughout this section. In a future version these assumptions will be replaced with the appropriate weaker conditions.

Much of this section will be spent laying the groundwork for the proofs of the following three propositions. In order of appearance we discuss, how vanishing lines behave under finite (co)limits, how vanishing lines behave under filterd (co)limits, sseqs and finally tensor products.

**Proposition 6.1.9.** *The full subcategory* $\mathrm{Syn}_E^{\omega,m=e}$ *of compact objects which admit a vanishing line of slope* $e$ *and the full subcategory* $\mathrm{Syn}_E^{\omega,\mu=\epsilon}$ *of compact objects which admit a finite-page vanishing line of slope* $\epsilon$ *are both thick* $\otimes$*-ideals in* $\mathrm{Syn}_E^{\omega}$.

**Proposition 6.1.10.** $X$ *admits a vanishing line with parameters* $(m, c)$ *if and only if* $X$ *is* $\tau$*-complete and* $C\tau \otimes X$ *admits the same vanishing line.*

**Proposition 6.1.11.** *There exists a function* $h_E : \mathbb{R} \to \mathbb{N} \cup \{\infty\}$ *such that a compact synthetic spectrum* $X$ *admits a finite-page vanishing line of slope* $\epsilon$ *if and only if* $n \geq h_E(\epsilon)$ *where* $n$ *is the type of* $X[\tau^{-1}]$*. We will refer to* $h_E$ *as the height-slope function.*

Although it might sound the most complicated, we can give the proof of Proposition 6.1.11 now (as long as we are willing to defer a key step step to **??** where we will treat the $tt$-geometry of $\mathrm{Syn}_E$ more seriously).

Proof. By Proposition 6.1.9 the full subcategory $\mathrm{Syn}_E^{\omega,\mu=\epsilon}$ is a thick $\otimes$-ideal of compact synthetic spectra. As explained in Example 6.1.2 know that $C\tau$ is an object of this category.

[4] As compared to [**BHS19**] the reader may note that the definitions have been simplified, the reader will find justification for this in the lemmas below.

[5] Unlike in [**BHS19**] we will not study the interaction of the functor $\nu$ with vanishing lines which streamlines the presentation substatially.

Applying **??** which classifies thick subcategories containing $C\tau$ based on the heights of objects after inverting $\tau$ we can conclude. □

The function $h_E$ essentially determines how chromatic phenomena behave in $\mathrm{Syn}_E$. In the next section we will compute the functions $h_{\mathrm{BP}}$ and $h_{\mathbb{F}_p}$ and comment on how this affects the structure of these categories.

Delve into what this function tells us and looks like in examples in a more substantial way.

6.1.1.1. *Finite (co)limits.*

In the stable setting finite (co)limits can be rewritten in terms of cofiber sequences. As a consequence the following lemma suffices for most applications.

**Lemma 6.1.12.** *Suppose we are given a cofiber sequence of synthetic spectra*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

*such that $A$ admits a finite-page vanishing line with $(m, c_1, r_1)$ and $C$ admits a finite-page vanishing line with $(m, c_2, r_2)$. Then, $B$ admits a finite-page vanishing line with $(m, \max(c_1 + r_2, c_2), r_1 + r_2)$.*

Proof. Suppose that $\alpha \in \pi_{k,s}(B)$ with $s > mk + \max(c_1 + r_2, c_2)$. Using the vanishing line for $C$ we learn $\tau^{r_2} g(\alpha) = 0$ and so we obtain a lift $\alpha'$ such that $f(\alpha') = \tau^{r_2}\alpha$. Using the vanishing line for $A$ we learn that $\tau^{r_1}\alpha' = 0$. Thus $\tau^{r_1+r_2}\alpha = 0$ as desired. □

The case of sums is special since the torsion bounds don't stack, so we record it separately.

**Lemma 6.1.13.** *If $A$ and $B$ admit finite-page vanishing lines with $(m, c_1, r_1)$ and $(m, c_2, r_2)$ respectively, then $A \oplus B$ admits a finite-page vanishing line with $(m, \max(c_1, c_2), \max(r_1, r_2))$.*

Proof. Clear. □

Although retracts and bigraded suspensions aren't examples of finite colimits we record how they affect vanishing lines here as well.

**Lemma 6.1.14.** *Suppose that $X$ has a finite-page vanishing line with $(m, c, r)$, then*

1. *any retract of $X$ has a finite-page vanishing line with $(m, c, r)$ and*
2. *$\Sigma^{k,s} X$ has a finite-page vanishing line with $(m, c - mk + s, r)$.*

Proof. Clear. □

With these lemmas in hand the proof of the following proposition is complete.

**Proposition 6.1.15.** *The full subcategory $\mathrm{Syn}_E^{\mu=\epsilon}$ of synthetic spectra which admit a finite-page vanishing line of slope $\epsilon$ is thick (closed under finite (co)limits and retracts) as well as closed under bigraded suspensions. Similarly, the full subcategory $\mathrm{Syn}_E^{m=e}$ is thick and closed under bigraded suspension.*

Note that the statement of this proposition is slightly different from Proposition 6.1.9 as we did not restrict to compact objects and have yet to consider tensor products.

6.1.1.2. *Filtered (co)limits.*
The behavoir of vanishing lines under filtered colimits is similarly easy to analyze.

**Lemma 6.1.16.** *A filtered colimit of synthetic spectra which each admit a finite-page vanishing line with $(m, c, r)$ admits a finite-page vanishing line with the same parameters.*

PROOF. $\pi_{**}(-)$ commutes with filtered colimits. □

The condition that each object in the filtered colimit has the same vanishing line parameters may seem overly restrictive. The trick here is that when applying this one can replace vanshing lines $(m, c_i, r_i)$ at an object $X_i$ with $c = \max_i(c_i)$ and $r = \max(r_i)$.

**Warning 6.1.17.** The preceeding lemma might lead the reader to think about finite-page vanishing lines with $r = \infty$ as the output of this lemma when the varying $r_i$ have no finite maximum. We warn them that this notion is not closed under cofiber sequences.

The version of Lemma 6.1.16 for cofiltered limits is substantially more complicated because of higher derived limits.

**Lemma 6.1.18.** *An $\mathbb{N}$-indexed limit of synthetic spectra which each admit a finite-page vanishing line with $(m, c, r)$ admits a finite-page vanishing line with $(m, c + 1 + m + r, 2r)$. If we assume that the $\varprojlim^1$ of the homotopy groups vanishes*[6]*, then this can be improved to $(m, c, r)$.*

PROOF. $\pi_{**}(-)$ doesn't quite commute with cofiltered limits, but we do have the Milnor sequene,

$$0 \to \varprojlim{}^1 \pi_{k+1,s-1}(X_\alpha) \to \pi_{k,s}\left(\varprojlim X_\alpha\right) \to \varprojlim \pi_{k,s}(X_\alpha) \to 0.$$

Examining how these short exact sequences behave under multiplication by $\tau$ as in the proof of Lemma 6.1.12 suffices to conclude. □

Since general colimits are built out of filtered colimits and finite colimits, the preceeding lemmas essentially determine the behavior of vanishing lines under all shapes of colimit. Our knowledge for limits is less complete, though practically speaking limits over uncountable diagrams are rare.

6.1.1.3. *Spectral sequences.*
The prototypical spectral sequence is just a an $\mathbb{N}$-inedexed diagram whose filtered colimit we would like to recover from its associated graded. We show that vanishing lines behave as expected in this setting[7].

**Lemma 6.1.19.** *Given a sequential colimit,*

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\infty$$

*let $F_i = X_i/X_{i-1}$ and $F_0 = X_0$. If the $F_i$ admit vanishing lines with $(m, c_i)$, then $X_\infty$ admits a vanishing line with $(m, \max\{c_i\})$.*

PROOF. After applying Lemma 6.1.12 $n$ times we learn that $X_n$ admits a vanishing line with $(m, \max_{0\le i\le n}(c_i))$. Using Lemma 6.1.16 to pass to $X_\infty$ we may conclude. □

Again $\varprojlim^1$ terms make the limit version more complicated.

[6]as is the case if the Mittag-Leffler condition is satisfied for example

[7]i.e. a vanishing line on the $E_1$-page yields a vanishing line for the object of interest.

**Lemma 6.1.20.** *Given a sequential limit,*

$$X_\infty \to \cdots \to X_2 \to X_1 \to X_0$$

*let* $F_i = \mathrm{fib}(X_i \to X_{i-1})$ *and* $F_0 = X_0$. *If the* $F_i$ *admit vanishing lines with* $(m, c_i)$, *then* $X_\infty$ *admits a vanishing line with* $(m, \max(c_i) + 1 + m)$. *If the maximum value of the* $c_i$ *occurs only finitely many times, then this can be improved to* $(m, \max(c_i))$.

Proof. As above we can apply Lemma 6.1.12 to conclude that $X_n$ admits a vanishing line with $(m, \max_{0\le i\le n}(c_i))$. Now we apply Lemma 6.1.18 in order to pass to $X_\infty$. In the situation where the maximum value occurs only finitely many times the relevant $\varprojlim^1$ term vanishes so we can just use the maximum of the $c_i$. ☐

At this point we are prepared to prove Proposition 6.1.10 by analyzing the $\tau$-bockstein tower.

**Proposition 6.1.21.** $X$ *admits a vanishing line with* $(m, c)$ *if and only if* $X$ *is* $\tau$*-complete and* $C\tau \otimes X$ *admits the same vanishing line.*

Proof. The proof is composed of two observations:

(1) If $X$ is not $\tau$-complete, then there exists some $k$ so that $\pi_{k,s}(X) \neq 0$ for all $s \gg 0$ (i.e. $X$ does not admit any vanishing lines).
(2) Upon applying lemma 6.1.20 to the $\tau$-completion tower we only have to analyze the associated graded, which is given by $\Sigma^{0,-n} C\tau \otimes X$ for $n \geq 0$.

☐

Since we encounter Bockstein towers quite frequently in this chapter we abstract the previous argument out so that we can reuse it.

**Lemma 6.1.22.** *Suppose we are given a synthetic spectrum* $X$, *a self-map* $b : \Sigma^{u,v} X \to X$ *and a compact localization* $L$ *which fits into a fiber sequence* $F \to \mathrm{Id} \to L$. *such that*

- $F\Sigma^{-|b|}\mathrm{cof}(b)$ *admits a vanishing line of slope* $m$ *and intercept* $c$,
- $X[b^{-1}]$ *is* $L$*-local,*
- $m \leq \frac{v}{u}$.

*Then,* $FX$ *admits a vanishing line of slope* $m$ *and intercept* $c + 1 + m^{-1}$

Proof. We begin by showing that $F\Sigma^{-n|b|}\mathrm{cof}(b^n)$ admits a vanishing line of slope $m$ and intercept $c$ for every $n$. We proceed by induction on $n$ with base case $n = 1$ given. Consider the cofiber sequence,

$$F\Sigma^{-|b|}\mathrm{cof}(b) \to F\Sigma^{-n|b|}\mathrm{cof}(b^n) \to F\Sigma^{-(n-1)|b|}\mathrm{cof}(b^{n-1}).$$

Then, using Lemma 6.1.12 and the inequality in the final bullet point we obtain the desired vanishing line.

Using the fact that $L$ commutes with colimits (and therefore $F$ commutes with colimits as well) we obtain an equivalence

$$F\mathrm{cof}(b^\infty) \simeq \varinjlim F\Sigma^{-n|b|}\mathrm{cof}(b^n).$$

Using Lemma 6.1.16 we now obtain a vanishing line of slope $m$ and intercept $c$ on $F\mathrm{cof}(b^\infty)$. Using our assumption that $X[b^{-1}]$ is $L$-local obtain an equivalence

$$FX \simeq \Sigma^{-1} F\mathrm{cof}(b^\infty)$$

from which we may conclude. ☐

6.1.1.4. *Tensor products.*

Unlike the other components of this section, there are general results about when a tensor product of synthetic spectra admits a vanishing line which are non-trivial to prove (and which depend on the fact that $E$ is one of $\mathbb{F}_p$ or BP). At the moment though we will stick with the theme of the section and only go after the low hanging fruit.

**Definition 6.1.23.** Recall that both $\mathrm{Syn}_{\mathrm{BP}}$ and $\mathrm{Syn}_{\mathbb{F}_p}$ admit a fiber functor to $\mathrm{Vect}^{\mathrm{gr}}_{\mathbb{F}_p}$, which in both cases is given by $C\tau \otimes \nu(\mathbb{F}_p) \otimes -$. This fiber functor is conservative on compact objects (but not conservative in general). We will denote this fiber functor by $h(-)$ for brevity. Recall that the rank of $h(X)$ records the minimum number of bigraded spheres needed to build $X$ and that there exists a cell structure which achieves this minimum. We will refer to the graded vector space $h(X)$ as the *homological location* of $X$. Since $h(X)$ can be regarded as a synthetic spectrum we will allow ourselves to speak of vanishing lines for it as well.

**Lemma 6.1.24.** *Suppose that $X$ admits a finite-page vanishing line with $(m, c_1, r)$ and $Y$ is compact. From compactness we know $h(Y)$ has rank finite rank and admits a vanishing line with $(m, c_2)$ for some $c_2$. Then, $X \otimes Y$ admits a finite-page vanishing line with parameters*

$$(m, c_1 + c_2 + r \cdot \mathrm{rank}(h(Y)), r \cdot \mathrm{rank}(h(Y)))\,.$$

Proof. Since $Y$ is compact we can use **??** to conclude that it can be built using cofiber sequences out of finitely many bigraded spheres indexed by $h(Y)$. Tensoring this construction of $Y$ with $X$ we obtain a construction of $X \otimes Y$ out of pieces which each admit finite page vanishing lines of slope $m$. Applying Lemma 6.1.14 and Lemma 6.1.12 each time we add a new cell now allows us to conclude. ☐

**Remark 6.1.25.** Note that we needed to assume that $Y$ was compact in this lemma since the torsion level depended on the number of cells of $Y$. In the next section we will show that in the case where $E$ is one of BP or $\mathbb{F}_p$ and we assume $X$ is compact as well, then a much more detailed analysis allows us to obtain good control over this torsion level.

The $r = 0$ case of this lemma doesn't depend on the compactness of $Y$ as directly since the there's no decay in the intercept. We record the following corollary which highlights how much better this situation behaves for later use.

**Corollary 6.1.26.** *Suppose that $X$ admits a vanishing line with $(m, c_1)$ and that $Y$ can be written as a filtered colimit of compact objects $Y_i$ for which $h(Y_i)$ admit a vanishing line with $(m, c_2)$. Then, $X \otimes Y$ admits a vanishing line with $(m, c_1 + c_2)$.*

Proof. We apply Lemma 6.1.24 to each $Y_i$ individually followed by Lemma 6.1.16 to handle the filtered colimit. ☐

Without all the parameters Lemma 6.1.24 becomes:

**Corollary 6.1.27.** *If $X$ admits a (finite-page) vanishing line of slope $m$ and $Y$ is compact, then $X \otimes Y$ admits a (finite-page) vanishing line of slope $m$ as well.*

In light of Proposition 6.1.15 this corollary finishes the proof of Proposition 6.1.9.

### 6.1.2. Slopes for BP and $\mathbb{F}_p$.

We began this section by defining an ascending, $\mathbb{R}$-indexed filtration of $\mathrm{Syn}_E^{\omega}$ by slopes

$$\cdots \subseteq \mathrm{Syn}_E^{\omega, m=[a,\infty]} \subseteq \mathrm{Syn}_E^{\omega, m=[b,\infty]} \subseteq \cdots \subseteq \mathrm{Syn}_E^{\omega, m=\infty} = \mathrm{Syn}_E^{\omega}.$$

In this subsection we determine the jumping locus of this filtration. This set is discrete and each jump point, $e$, is characterized by the existence of a synthetic spectrum which admits a vanishing line of slope $e$, but does not admit a vanishing line of any smaller slope. In each case these vanishing lines arise for what is essentially the simplest possible reason: Whenever a compact $X$ has a vanishing line of slope exactly $e$ there is a non-nilpotent self-map of $X$ which acts parallel to this vanishing line.

**Notation 6.1.28.** We let $m(i,j)$ denote the slope of the May generator $b_{i,j}$ and $v(n)$ denote the slope of the $n^{\text{th}}$ chromatic periodicity. More specifically this means

$$v(n) \coloneqq (2p^n - 2)^{-1} \text{ and } m(i,j) \coloneqq \begin{cases} (2^{i+j} - 2^j - 1)^{-1} & \text{if } p = 2 \\ (p^{i+j+1} - p^{j+1} - 1)^{-1} & \text{if } p \neq 2 \end{cases}.$$

We will also use the compact notation $\mathrm{Syn}_E^{m(i,j)}$ as a replacement for the more cumbersome $\mathrm{Syn}_E^{m=m(i,j)}$ (and similar with $v(n)$).

6.1.2.1. *Algebraic periodicity.*

6.1.2.2. *Synthetic periodicity.*

6.1.2.3. *Slope-Moore objects.*

### 6.1.3. Slope localizations.

In the previous section we explained how using slope-Moore objects we can calculate the bigraded homotopy groups of synthetic spectra above a line as the output of a finite number of Bockstein sseqs. For easy applications this approach can be useful, but because the initial constructions lacked functoriality the whole thing is a house of cards. In this section we pass to a more rigid and categorical perspective.

We would like to know in what sense the study of homotopy groups above a line can be replaced by the study of a *slope localization*. This requires several developments. In this section we will construct localizations associated to each (allowable) slope and investigate their basic behavior. These localizations can be assembled into a tower and in the section following this we will investigate the convergence properties of this tower.

6.1.3.1. *Constructing slope localizations.*

Each slope localization $L_{m(i,j}$ sits in a cofiber sequence with its associated acyclicization functor $F_{m(i,j)}$

$$F_{m(i,j)} \to \mathrm{Id} \to L_{m(i,j)}.$$

6.1.3.2. *Mono-slope localizations.*

6.1.3.3. *Examples of slope localizations.*

In this version the reader can treat the next couple examples as the definitions of these localizations.

**Example 6.1.29.** The localization $L_{v(0)}$ is the compact localization with kernel generated by $\mathbb{S}/\widetilde{p}$. This localization has a more concrete description as inverting $\widetilde{p}$.

**Example 6.1.30.** The localization $L_{v(1)}$ is the compact localization with kernel generated by $\nu V(1)$ (at odd primes). From the description of $\nu V(1)$ as $\mathrm{cof}\left(\mathbb{S}/\widetilde{p} \xrightarrow{\widetilde{v}_1} \mathbb{S}/\widetilde{p}\right)$ we obtain a description of $L_{v(1)}\,\mathbb{S}/\widetilde{p}$ as $\mathbb{S}/\widetilde{p}[\widetilde{v}_1^{-1}]$.

**Example 6.1.31.** The localization $L_{=v(1)}$ is the Bousfield localization with respect to $\mathbb{S}/\widetilde{p}[\widetilde{v}_1^{-1}]$.

**Example 6.1.32.** The localization $L_{m(1,0)}$ is the compact localization with kernel generated by $\nu V(1) \otimes \mathbb{S}/\beta_1$.

**Lemma 6.1.33.**
(1) *$L_{v(0)}$ can also be described as Bousfield localization with respect to $\mathbb{S}[\widetilde{p}^{-1}]$.*
(2) *$L_{v(1)}$ can also be described as Bousfield localization with respect to $\mathbb{S}[\widetilde{p}^{-1}] \oplus \mathbb{S}/\widetilde{p}[\widetilde{v}_1^{-1}]$.*
(3) *$L_{m(1,0)}$ can also be described as Bousfield localization with respect to $\mathbb{S}[\widetilde{p}^{-1}] \oplus \mathbb{S}/\widetilde{p}[\widetilde{v}_1^{-1}] \oplus \mathbb{S}[\beta_1^{-1}]$.*

PROOF. In order to prove the lemma we just need to compute the kernel of each Bousfield localization and show that it is generated by the appropriate object. All three cases are quite similar, so we only give details for (2).

Suppose that $X \otimes (\mathbb{S}[\widetilde{p}^{-1}] \oplus \mathbb{S}/\widetilde{p}[\widetilde{v}_1^{-1}]) = 0$. Using the vanishing from the first factor we learn that for any map $f : Y \to X$ from a compact object $Y$ the composite map

$$\Sigma^{0,N} Y \xrightarrow{\widetilde{p}^N} Y \xrightarrow{f} X$$

is null. Consequently we can factor $f$ as $Y \to Y \otimes \mathbb{S}/\widetilde{p}^N \xrightarrow{g} X$. Using the vanishing from the second factor we learn that for any map $g : Y \otimes \mathbb{S}/\widetilde{p}^N \to X$ from a compact object $Y$ the composite map

$$\Sigma^{0,M} Y \otimes \mathbb{S}/\widetilde{p}^N \xrightarrow{\widetilde{v}_1^M} Y \otimes \mathbb{S}/\widetilde{p}^N \xrightarrow{g} X$$

is null. Consequently we can factor $g$ as $Y \otimes \mathbb{S}/\widetilde{p}^N \to Y \otimes \mathbb{S}/(\widetilde{p}^N, \widetilde{v}_1^M) \xrightarrow{h} X$. Since $\mathbb{S}/(\widetilde{p}^N, \widetilde{v}_1^M)$ is in the thick $\otimes$-ideal generated by $\nu V(1)$ this is enough to conclude that $X$ is in the localizing subcategory generated by $\nu V(1)$. □

Using Lemma 6.1.33 and Lemma 1.2.1 we obtain fracture squares such as

$$\begin{array}{ccc} L_{m(1,0)}(-) & \longrightarrow & (-)[\beta_1^{-1}] \\ \downarrow & & \downarrow \\ L_{v(1)}(-) & \longrightarrow & L_{v(1)}(-)[\beta_1^{-1}] \end{array}$$

which let us understand $L_{m(1,0)}$ in terms of simpler localizations.

6.1.3.4. *The slope tower.*

6.1.3.5. *Slope fracture squares.*

#### 6.1.4. Chromatic slope convergence.

#### 6.1.5. Residue fields and an $\mathbb{F}_p$-synthetic telescope conjecture.

## 6.2. *tt*-Geometry

## 6.3. Computations in localizations of the Adams spectral sequence

In the previous section we identified the slope localizations of the category of $\mathbb{F}_p$-synthetic spectra and described the general properties shared by these localization. Through **??**, the study of the Adams sseq above a line of slope $m$ is transformed into a discrete sequence of steps involving the computation of a full-plane localized sseq followed by a re-assembly step (mediated by fracture squares **??**). **??** indicates the converse is also true: if one understands the Adams sseq above a line of slope $m$, then one *implicitly also understands* both the slope localizations for $m' > m$ and the way in which these re-assemble.

In this section we carry out this process for some of the larger slopes. Although we summarize the contents of each subsection below, this outline focuses only on the results obtained rather than the techniques developed. The true purpose of this section, however, is to showcase these computational techniques. For this reason we are less than systematic in our study and proceed only as far as is necessary for the purpose of demonstration.

### 6.3.1. The $\widetilde{p}$-local category.

The largest slope is $m = \infty$ and this corresponds to the $\widetilde{p}$-local category. A complete analysis of this category is relatively easy and we will use this as an opportunity to illustrate the basic pattern later subsections will follow. The content of this section is essentially equivalent to that of [**Ada66b**, Section 2].

**Lemma 6.3.1.** *The map $\mathbb{S} \to \nu\,\mathbb{Z}_p$ is a $\widetilde{p}$-local equivalence.*

Proof. Let $X_i$ denote the $i^{\text{th}}$ stage in an integral skeleton for $\mathbb{Z}_p$ (see **??** for the discussion regarding integral skeletons). Since the map $X_i \to X_{i+1}$ is injective on $\mathbb{F}_p$-homology we find that $\nu\,\mathbb{Z}_p$ has a filtration with associated graded $\nu(X_i/X_{i-1})$. It will suffice for us to show that for $i > 0$ this associated graded is $\widetilde{p}$-locally trivial.

These quotients will be a sum of mod $p$ Moore spaces in degrees where the Bockstein on the $\mathbb{F}_p$ homology is exact (and since $\nu(\mathbb{S}/p) \cong \mathbb{S}/\widetilde{p}$ this will be sufficient). The $\mathbb{F}_p$-homology of $\mathbb{Z}_p$ is given by

$$\mathcal{A}_{\mathbb{Z}} = \mathbb{F}_p[\zeta_1^2, \zeta_2, \dots] \subset \mathcal{A} \text{ at } p = 2$$
$$\mathcal{A}_{\mathbb{Z}} = \mathbb{F}_p[\xi_1, \xi_2, \dots]\langle \tau_1, \dots \rangle \subset \mathcal{A} \text{ otherwise}$$

and we can read off from the coproduct that $\beta$ acts as a derivation with $\beta(\zeta_i) = \zeta_{i-1}^2$ and $\beta(\tau_i) = \xi_i$. In degrees 1 and larger this is exact, as desired. $\square$

In [location] we calculated that

$$\pi_{0,*}\,\mathbb{S} \cong \pi_{0,*}\nu\,\mathbb{Z}_p \cong \mathbb{Z}_p[\widetilde{p}, \tau]/(\widetilde{p}\tau = p).$$

Since the homotopy of $\nu\,\mathbb{Z}_p$ vanishes in positive topological degrees we can now compute the bigraded homotopy of the $\widetilde{p}$-local sphere:

$$\pi_{**}\,\mathbb{S}[\widetilde{p}^{-1}] \cong \mathbb{Z}_p[\widetilde{p}^{\pm 1}]$$

with $\tau = p \cdot \widetilde{p}^{-1}$.

**Proposition 6.3.2.** *There is an equivalence of stable psmc*

$$\mathrm{Syn}_{\mathbb{F}_p}[\widetilde{p}^{-1}] \cong \mathrm{Ab}_{(p)}.$$

*Under this equivalence $C\tau$ is sent to $\mathbb{F}_p$.*

As a consequence of this proposition can conclude that any compact object in the $\widetilde{p}$-local category is a sum of (suspensions of) copies of $\mathbb{1}$ and $\mathbb{1}/p^k$.

Proof. We will prove this proposition by showing that $\mathrm{Syn}_{\mathbb{F}_p}[\widetilde{p}^{-1}]$ is affine over Sp and then computing the spectrum of endomorphisms of the unit. For affineness, using **??** it suffices to note that $\mathrm{Syn}_{\mathbb{F}_p}$ is generated as a stable category by $\{\mathbb{S}^{0,s}\}$ and in the $\widetilde{p}$-local category we have equivalences $\widetilde{p} : \mathbb{S}^{0,s} \to \mathbb{S}^{0,s-1}$.

Since there is an essentially unique commutative algebra with the desired homotopy groups we just need to use our computation that $\pi_{0,0}\,\mathbb{S}[\widetilde{p}^{-1}] \cong \mathbb{Z}_{(p)}$ and that the higher degree groups all vanish. □

We now translate this information backwards into a description of the Adams sseq above a line of slope $q^{-1}$. From **??** we already knew that if $X$ is compact, then the fiber of the $\widetilde{p}$-localization map, $F_\infty X$, admits a vanishing line of slope $q^{-1}$. We sharpen this information in two ways. First we calculate an explicit intercept for this vanishing line in the case of the sphere.

**Lemma 6.3.3** (Adams)**.** *The fiber of $\widetilde{p}$-localization on the sphere $F_{v(0)}\,\mathbb{S}$ admits a vanishing line of slope $q^{-1}$ and intercept $3/2$ ($2q^{-1}$ for $p$ odd).*

PROOF. Using Lemma 6.1.22 is will suffice to show that $\mathbb{S}/\widetilde{p}$ has a vanishing line of slope $q^{-1}$ and intercept 1 (intercept $q^{-1}$ for $p$ odd). Using Proposition 6.1.21 we reduce this to an Ext computation.

At odd primes we have the change-of-rings isomorphism

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{1}/q_0) \cong \mathrm{Ext}_{\mathcal{A}_{\mathbb{Z}}}(\mathbb{1}).$$

The desired vanishing line is now present on the first page of the May sseq for $\mathcal{A}_{\mathbb{Z}}$.

At $p = 2$ our argument is more elaborate. Let $Y$ denote $\mathrm{Cof}(\widetilde{2}) \otimes \mathrm{Cof}(\eta)$. Another May sseq argument lets us conclude that $Y$ has a vanishing line of slope $1/2$ and intercept 0 . Since $\eta^3 = \widetilde{2}^2\nu$, we learn that $\mathrm{Cof}(\widetilde{2}) \oplus \Sigma^{4,2}\mathrm{Cof}(\widetilde{2})$ has a three step filtration with associated graded $Y$, $\Sigma^{1,1}Y$, $\Sigma^{2,2}Y$. Applying lemmas 6.1.14 and 6.1.12 we conclude that $\mathrm{Cof}(\widetilde{2})$ admits a vanishing line of slope $1/2$ and intercept 1. □

Add more details in a later version.

#### 6.3.2. The $\widetilde{v_1}$-local category at odd primes.

In this, mostly expository, section we study the second slope localization which corresponds to the $\widetilde{v_1}$-local subcategory of $\mathrm{Syn}_{\mathbb{F}_p}$. Although this is only the second slope localization it presents several new challenges not present in the $\widetilde{p}$-local category.

6.3.2.1. *The Moore spectrum.*

Since the Moore spectrum $\mathbb{S}/\widetilde{p}$ is also a slope Moore spectrum the $\widetilde{v_1}$-localization of this object *is* computed by inverting a self-map. We begin by recasting Miller's verification of the height 1 telescope conjecture at odd primes as a computation of the homotopy groups of $\mathbb{S}/\widetilde{p}[\widetilde{v_1}^{-1}]$.

**Theorem 6.3.4** (Miller, [Mil81])**.** *Above a line of slope $m(1,0)$ the Adams sseq for* $\mathrm{Cof}(p)$ *collapses at $E_3$ where it is isomorphic to $\mathbb{F}_p[q_1^{\pm 1}]\langle h_{1,0}\rangle$.*

In the synthetic language Miller's result becomes the following:

**Proposition 6.3.5.** *The homotopy groups of $(\mathbb{S}/\widetilde{p})[\widetilde{v_1}^{-1}]$ are given by $\mathbb{F}_p[\tau, \widetilde{v_1}^{\pm}]\langle\alpha_1\rangle \oplus T$ where every element of $T$ is simple $\tau$-torsion.*

We will recast Miller's proof as an application of the $\nu BP$-based Adams sseq together with a rather useful counting lemma.

6.3.2.2. *The $\nu$BP-based Adams sseq.*

6.3.2.3. *The $\widetilde{v}_1$-local sphere.*

In this section we will recast the computations from Michael Andrews' thesis [**And15**] as the computation of the homotopy groups of the $\widetilde{v}_1$-local sphere. We will not reproduce the computations ourselves, instead only providing the necessary material to link the two perspectives. The key translation between these perspectives is that the sseq which Andrews refers to as the MASS-$\infty$ can be identified with the $\tau$-Bockstein sseq for $F_{v(0)}L_{v(1)}\mathbb{S}$ (indexed so that the homotopy groups of the cofiber of $\tau$ are the $E_2$-page).

**Notation 6.3.6.** Let $[n]$ denote the $q$-analog of $n$ evaluated at $p$, i.e. $[n] = \frac{p^n-1}{p-1}$.

**Theorem 6.3.7** ([**And15**])**.** *The $E_3$-page of the $\tau$-Bockstein sseq for $F_{v(0)}L_{v(1)}\mathbb{S}$ has an $\mathbb{F}_p$-basis consisting of classes*

- $q_0^v$ *for* $v < 0$ *in degree* $(-1,1) + |\widetilde{p}|v$,
- $q_0^v q_1^{kp^{n-1}}$ *for* $n \geq 1$, $p \not| k$ *and* $-1-[n-1] \leq v < 0$ *in degree* $(-1,1) + |\widetilde{p}|\cdot v + |\widetilde{v}_1|\cdot kp^{n-1}$,
- $q_0^v q_1^{kp^n}\epsilon_n$ *for* $n \geq 1$, $p|k$ *and* $1-p^n \leq v < 0$ *in degree* $(-2, 2-[n]) + |\widetilde{p}|\cdot v + |\widetilde{v}_1|\cdot kp^n$.

*The differentials are then determined by the following two pieces of information: the classes $q_0^v$ are permanent cycles and $q_0^v q_1^{kp^n}\epsilon_n$ is hit by a differential of length $|k|_p + 2$ off of a class of the form $q_0^? q_1^?$. This leaves an $E_\infty$-page with a basis consisting of the classes*

- $q_0^v$ *for* $v < 0$,
- $q_0^v q_1^{kp^{n-1}}$ *for* $n \geq 1$, $p \not| k$ *and* $-n \leq v < 0$,
- $q_0^v \epsilon_n$ *for* $n \geq 1$ *and* $1-p^n \leq v < 0$.

**Remark 6.3.8.** It is notable that the $E_3$-page described in Theorem 6.3.7 is sufficiently sparse that there is at most a single copy of $\mathbb{F}_p$ in each bidegree.

**Remark 6.3.9.** From the $\widetilde{p}$-local computations from the previous section and the fact that $L_{v(0)}L_{v(1)}\mathbb{S} \simeq L_{v(0)}\mathbb{S}$ we can extract a description of the bigraded homotopy groups $\pi_{n,s}L_{v(1)}\mathbb{S}$ for $n > 0$ from Theorem 6.3.7

**Proposition 6.3.10.** *Suppose that $x \in \pi_{n,s}L_{v(1)}\mathbb{S})$ and it maps to zero in the classical $L_1$-local sphere. Then,*

(1) *If* $n \not\equiv -2 \pmod{2p-2}$, *then* $\tau x = 0$.
(2) $x$ *is* $\tau^c$*-torsion where* $c = \max\{1, |n+2|_p\}$.
(3) *If* $n \geq 0$ *and* $s \leq \frac{p-2}{2(p-1)^2}(n+2) + 2$, *then* $\tau x = 0$.

Proof. (1) follows from the sparsity of the $E_3$-page in Theorem 6.3.7. (2) follows from the description of the lengths of the differentials in Theorem 6.3.7. (3) is similar, but more combinatorially involved. We can read off from the $E_3$-page in Theorem 6.3.7 that in topological degree $jp^r(2p-2)-2$ where $p \nmid j$ the $\tau^2$-torsion class of minimal Adams filtration is

$$q_0^{1-p^{r-1}} q_1^{jp^r}\epsilon_{r-1} \quad \text{in degree} \quad (jp^r(2p-2)-2,\ jp^r + 3 - [r])$$

Noting that the line specificed in (3) passes below these classes we obtain the desired result. □

### 6.3.3. The $\widetilde{v}_1$-local category at $p = 2$.

**6.3.4. The slope $m(1,0)$-localization at odd primes.**
In this section we study the slope $m(1,0)$-localization at odd primes. The associated self-map which runs paralell to this slope is $\beta_1$ and the fact that this class was defined in the sphere is a key feature that makes studying this localization an approachable problem.

In the first subsection we will study the $\beta_1$-inverted category. In the second we combine our understanding of the $\beta_1$ inverted category with our understanding of the $\widetilde{v}$-local category to study the slope $m(1,0)$-local category. In the third subsection we analyze the range in $\pi_{**}\,\mathbb{S}$ agrees with the bigraded homotopy groups of $L_{m(1,0)}\,\mathbb{S}$.

6.3.4.1. *The $\beta_1$-inverted category.*
In this version we will only go after the most basic piece of information about the $\beta_1$-inverted category.

**Notation 6.3.11.** Let $A_3 = 5$ and $A_p = 2(p-1)^2$ for $p \geq 5$.

**Lemma 6.3.12.** *The bigraded homotopy groups of $\mathbb{S}[\beta_1^{-1}]$ are $\tau^{A_p}$-torsion.*

Proof. It will suffice to show that $\tau^{A_p}\beta_1^N = 0$ for some $N \gg 0$. Using the fact that the comparison functor $\mathrm{Syn}_{\mathrm{BP}} \to \mathrm{Syn}_{\mathbb{F}_p}$ sends $\beta_1$ to $\beta_1$ (the implicit claim here is that $\beta_1$ has equal Adams and Adams-Novikov filtrations of 2) it will suffice to prove this relation in $\mathrm{Syn}_{\mathrm{BP}}$. In this form this claim was prove in [BHS19, Proof of Theorem 12.2], though this itself was an easy consequence of the bound on the length of the Adams-Novikov differential killing $\beta_1^{p^2-p+1}$ as discussed in Ravenel's Green book shortly after the statement of Theorem 7.6.1. The proof that $\beta_1^{p^2-p+1} = 0$ is due to Toda.

At the prime 3 we will make an improvement on the argument above. From the 3-primary Adams spectral sequence calculations of Oka [Oka71, Oka72] we can read off that $\pi_{60,12}\,\mathbb{S} \cong \mathbb{F}_p\{\beta_1^6\}$. Since $b_1^6$ is hit by a $d_6$ differential in the $\mathbb{F}_3$-Adams spectral sequence some (and therefore every) element of $\pi_{60,12}\,\mathbb{S}$ is $\tau^5$-torsion. □

6.3.4.2. *The $m(1,0)$-localization.*
In order to study the $m(1,0)$-local category we will use a fracture square to combine our previous knowledge about the $v(1)$-local category with what we know about the $\beta_1$-inverted category.

**Lemma 6.3.13.** *The bigraded homotopy groups of $L_{v(1)}\,\mathbb{S}[\beta_1^{-1}]$ are all simple $\tau$-torsion.*

Proof. Consider the composition

$$\mathbb{S} \to L_{v(1)}\,\mathbb{S} \to L_{v(1)}\,\mathbb{S}[\beta_1^{-1}],$$

we will show that the image of $\beta_1$ in $L_{v(1)}\,\mathbb{S}$ is simple $\tau$-torsion. First we note that $L_{v(1)}\,\mathbb{S}$ only has $\tau$-torsion-free elements in topological degrees congruent to $-1$ and $0$ mod $2p-2$. In particular, we learn that the image of $\beta_1$ is $\tau^{\infty}$-torsion. By proposition 6.3.10(2) we may now conclude that $\beta_1 \in \pi_{10,2}(L_{v(1)}\,\mathbb{S})$ is simple $\tau$-torsion. □

**Definition 6.3.14.** Let $B(n,s)$ denote the $\tau$-torsion-order of $\pi_{n,s}L_{v(1)}\,\mathbb{S}$.

**Proposition 6.3.15.** *The $\tau$-torsion-order of $\pi_{n,s}L_{m(1,0)}\,\mathbb{S}$ is at most $1+\max(A_p, B(n,s))$.*

Proof. Consider the fracture square expressing the $L_{m(1,0)}$-localization in terms of the $L_{v(1)}$-localization and inverting $\beta_1$

$$\begin{array}{ccc} L_{m(1,0)}\,\mathbb{S} & \longrightarrow & L_{v(1)}\,\mathbb{S} \\ \downarrow & & \downarrow \\ \mathbb{S}[\beta_1^{-1}] & \longrightarrow & L_{v(1)}\,\mathbb{S}[\beta_1^{-1}]. \end{array}$$

Given a $\tau^\infty$-torsion element $x \in \pi_{n,s} L_{m(1,0)}\,\mathbb{S}$ which maps to $y \in \pi_{n,s} L_{v(1)}\,\mathbb{S}$ and $z \in \pi_{n,s}\,\mathbb{S}[\beta_1^{-1}]$ we argue as follows: From Lemma 6.3.12 we know that $\tau^{A_p} z = 0$. By definition we know that $\tau^{B(n,s)} y = 0$. Therefore, $\tau^{\max(A_p, B(n,s))} x$ lifts to some element $w$ in the homotopy of $\Sigma^{-1} L_{v(1)}\,\mathbb{S}[\beta_1^{-1}]$. Since $\tau w = 0$ by Lemma 6.3.13 we learn that $\tau^{1+\max(A_p,B(n,s))} x = 0$ as desired. □

6.3.4.3. *The region of convergence.*

In this subsection we analyze the size of the region in which the bigraded homotopy groups of the sphere agree with those of the $m(1,0)$-local sphere. With more complete knowledge of the $m(1,0)$-local sphere this would allows us to understand the Adams spectral sequence for the sphere above a line of slope $v(2)$.

**Proposition 6.3.16.** *The synthetic spectrum $F_{m(1,0)}\,\mathbb{S}$ has a vanishing line of slope $v(2)$ and intercept $1 + 4v(2)$.*

In order to prove proposition 6.3.16 we begin with the following lemma which will allow us to reduce the proof to analyzing $\nu V(1) \otimes \mathbb{S}/\beta_1$.

**Lemma 6.3.17.** *Suppose that $\nu V(1) \otimes \mathbb{S}/\beta_1$ has a vanishing line of slope $v(2)$ and intercept $c$.*

1. *$\Sigma^{-|\widetilde{p}|-|\widetilde{v}_1|-|\widetilde{\beta_1}|} F_{m(1,0)}(\nu V(1) \otimes \mathbb{S}/\beta_1)$ has a vanishing line of slope $v(2)$ and intercept $c - 3 - 2v(2)$.*
2. *$\Sigma^{-|\widetilde{p}|-|\widetilde{v}_1|} F_{m(1,0)} \nu V(1)$ has a vanishing line of slope $v(2)$ and intercept $c - 2 - v(2)$.*
3. *$\Sigma^{-|\widetilde{p}|} F_{m(1,0)}\,\mathbb{S}/\widetilde{p}$ has a vanishing line of slope $v(2)$ and intercept $c - 1$.*
4. *$F_{m(1,0)}\,\mathbb{S}$ has a vanishing line of slope $v(2)$ and intercept $c + v(2)$.*

Proof. First, we note that

$$L_{m(1,0)}\,(\nu V(1) \otimes \mathbb{S}/\beta_1) \simeq 0.$$

Therefore we can read (1) off by tranlating the given vanishing line appropriately,

$$c + (-1) + (-1 + (2p-2)v(2)) + (-2 + (2p^2 - 2p - 2)v(2)) = c - 3 - 2v(2).$$

Second, we apply lemma 6.1.22 with $X = \Sigma^{-|\widetilde{p}|-|\widetilde{v}_1|} \nu v(1)$, $b = \beta_1$ and $L = L_{m(1,0)}$. In this case $\nu V(1)[\beta_1^{-1}]$ is $L_{m(1,0)}$-local since it is $\beta_1$-local. Third, we apply lemma 6.1.22 with $X = \Sigma^{-|\widetilde{p}|}\,\mathbb{S}/\widetilde{p}$ and $b = \widetilde{v}_1$. In this case $(\mathbb{S}/\widetilde{p})[\widetilde{v}_1^{-1}]$ is $L_{m(1,0)}$-local since it is $L_{v(1)}$-local. Finally, we apply lemma 6.1.22 with $X = \mathbb{S}$ and $b = \widetilde{p}$. In this case $\mathbb{S}[\widetilde{p}^{-1}]$ is $L_{m(1,0)}$-local since it is $\widetilde{p}$-local.

□

**Lemma 6.3.18.** *The synthetic spectrum $\nu V(1) \otimes \mathbb{S}/\beta_1$ admits a vanishing line of slope $v(2)$ and intercept $1 + 3v(2)$.*

Proof. As above, we note that $\nu V(1) \otimes \mathbb{S}/\beta_1$ is $L_{m(1,0)}$-acyclic. Therefore, by the results of [location] it admits a vanishing line of slope $v(2)$. The main content of this lemma lies in finding a good intercept for this vanishing line.

We will begin by analyzing the spectrum $T(0)_{(1)}$ from [**Rav86**, Example 7.1.17] and its relation to $\mathbb{S}/\beta_1$. Taking $m = 0$ and $h = p - 1$ in [**Rav86**, 7.1.11] we learn that there is a sequence of inclusions of $\mathrm{BP}_*\mathrm{BP}$-comodules

$$\mathrm{BP}_*(T(0)_{(1)}) \cong \mathrm{BP}_*\{1, t_1, \ldots, t_1^{p-1}\} \subset \mathrm{BP}_*[t_1] \subset \mathrm{BP}_*[t_1, t_2, \ldots] \cong \mathrm{BP}_*\mathrm{BP}.$$

From this we learn that there is a sequence of inclusions of $\mathcal{A}$-comodules

$$(H\,\mathbb{F}_p)_*T(0)_{(1)} \cong \mathbb{F}_p\{1, \xi_1, \ldots, \xi_1^{p-1}\} \subset \mathbb{F}_p[\xi_1] \subset \mathcal{A}.$$

In section 9.3 of [**Rav92**] $T(0)_{(1)}$ is called $G_1$ instead and Ravenel shows that $\mathbb{S}^0$ is in the thick subcategory generated by $G_1$. We will break with both of these conventions and refer to this spectrum as $X_p$.

Our next step is finding an explicit intercept for a vanishing line for $\nu V(1) \otimes \nu X_p$ of the desired slope. By Proposition 6.1.10 it will suffice to prove that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p[\xi_1, \ldots, \xi_1^{p-1}] \otimes E(\tau_0, \tau_1))$$

satisfies the same vanishing criterion. The May $\mathrm{E}_1$-term for this Ext group is given by modding out by $b_{1,0}, h_{1,0}, v_0$ and $v_1$ on the May $\mathrm{E}_1$-term for the sphere. Thus, we can read off that $\nu V(1) \otimes \nu X_p$ has a vanishing line of slope $v(2)$ and intercept $v(2)2p$.

Now we must pass back to $\mathbb{S}/\beta_1$ from $X_p$. Classically there are cofiber sequences

$$\mathbb{S}^0 \to X_p \xrightarrow{j} \Sigma^q X_{p-1} \quad \text{and} \quad X_{p-1} \xrightarrow{i} X_p \to \mathbb{S}^{q(p-1)}$$

where the maps $\mathbb{S}^0 \to X_p$ and $X_p \to \mathbb{S}^{q(p-1)}$ are the inclusion of the bottom cell and the pinch onto the top cell respectively such that $\mathrm{fib}(ij) \simeq \mathbb{S}/\beta_1$ (see Section 9.3 of [**Rav92**]). Since both of these cofiber sequences are short exact on homology, they remain cofiber sequences after applying $\nu$. We expand the equation $\nu(ij) = \nu(j) \circ \nu(i)$ into the a diagram where each row and column is a cofiber sequence in the way shown below,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \nu X_p & = & \nu X_p & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \nu(j)} & & \downarrow{\scriptstyle \nu(ij)} & & \downarrow \\
\mathbb{S}^{qp-1,qp} & \longrightarrow & \Sigma^{q,q}\nu X_{p-1} & \xrightarrow{\nu(i)} & \Sigma^{q,q}\nu X_p & \longrightarrow & \mathbb{S}^{qp,qp} \\
\| & & \downarrow & & \downarrow & & \| \\
\mathbb{S}^{qp-1,qp} & \longrightarrow & \mathbb{S}^{1,0} & \longrightarrow & \mathrm{cof}(\nu(ij)) & \longrightarrow & \mathbb{S}^{qp,qp}
\end{array}$$

and this diagram lets us identify this synthetic construction as producing $\mathbb{S}/\beta_1$ as well since the construction we have make is compatible with inverting $\tau$ and $\pi_{qp-2,qp}(\mathbb{S}^{0,0}) \cong \mathbb{F}_p\{\tilde{\beta}_1\}$.

Rotating the third vertical cofiber sequence above we obtain a cofiber sequence

$$\Sigma^{2p-3,2p-2}\nu X_p \to \mathbb{S}/\beta_1 \to \nu X_p.$$

Tensoring this cofiber sequence with $\nu V(1)$ and using the vanishing line established above for $\nu V(1) \otimes \nu X_p$ allows us to conclude that $\mathbb{S}/\beta_1$ has a vanishing line of slope $v(2)$ and intercept $v(2)2p + (1 - (2p-3)v(2))$. □

Proof (of Proposition 6.3.16). Plugging lemma 6.3.18 into lemma 6.3.17 we obtian the desired vanishing line. □

**Definition 6.3.19.** Let $\Gamma_p(n)$ denote the minimal $s$ such that every $\alpha \in \pi_{n,s+1}\mathbb{S}$ is either $\tau$-torsion or detected $K(1)$-locally after inverting $\tau$.

**Proposition 6.3.20.** *For $p \neq 2$, there is a line of slope $v(2)$ in the Adams spectral sequence such that every class above it is in the Image of $J$. More precisely,*

$$\Gamma_3(n) \leq \frac{1}{16}n + 7 + \frac{1}{4}$$
$$\Gamma_p(n) \leq v(2)n + 2p^2 - 4p + 4 + 4v(2) \qquad \text{for } p \geq 5.$$

Proof. Consider the cofiber sequence associated to the $L_{m(1,0)}$-localization of the sphere

$$F_{m(1,0)}\,\mathbb{S} \to \mathbb{S} \to L_{m(1,0)}\,\mathbb{S}\,.$$

Suppose we are given an element $x \in \pi_{n,s}\,\mathbb{S}$ whose image in the classical $K(1)$-local sphere is zero. Then, since $L_{m(1,0)}\,\mathbb{S}[\tau^{-1}] \simeq L_1\,\mathbb{S}$ we may conclude that it maps to $\tau^\infty$-torsion in $L_{m(1,0)}\,\mathbb{S}$. Note that since $x$ started in the sphere we may assume that $n \geq 0$. To prove the proposition we will use Proposition 6.3.15 to bound the $\tau$-torsion order of $x$ in the $m(1,0)$-localization and Proposition 6.3.16 to argue that this is enough to force $x$ to be $\tau^\infty$-torsion in the sphere.

We would like apply Proposition 6.3.15 to conclude that the image of $\tau^{1+A_p}x$ in $L_{m(1,0)}\,\mathbb{S}$ is zero. For $n < (2p-2)p^{A_p+1} - 2$ we have that $B(n,s) \leq A_p$ by Proposition 6.3.10(2) so we may conclude. When $n \geq (2p-2)p^{A_p+1} - 2$ we replace $x$ by $\tau^N x$ for the minimal $N$ such that $|\tau^N x|$ is on or below the line

$$s \leq \frac{p-2}{2(p-1)^2}(n+2) + 2.$$

We will deal with the complications this replacement incurs at the end of the proof. By proposition 6.3.10 the torsion bound $B(n,s)$ is 1 after such a replacement so again we may appeal to Proposition 6.3.15.

Since the image of $\tau^{1+A_p}x$ maps to zero in $L_{m(1,0)}\,\mathbb{S}$ it lifts to the fiber $F_{m(1,0)}\,\mathbb{S}$. Here we may use Proposition 6.3.16 to conclude that the lift of $\tau^{1+A_p}x$ is zero as long as

$$s - 1 - A_p > v(2)n + 1 + 4v(2).$$

Unwinding the replacement of $x$ by a $\tau$-power of $x$ made above we learn that this argument is valid when

$$\left(\frac{p-2}{2(p-1)^2}(n+2) + 1\right) - 1 - A_p > v(2)n + 1 + 4v(2)$$

Simplifying and replacing the factor in front of $(n+2)$ with a possibly smaller one it will suffice to check that $v(2)n \geq A_p + 2 + 4v(2)$ when $n \geq (2p-2)p^{A_p+1} - 2$ which is clear. ☐

#### 6.3.5. The $\bar{\kappa}$-local category.

#### 6.3.6. Consequences for the sphere.

### 6.4. Computations in localizations of the Adams–Novikov spectral sequence

### 6.5. $\mathbb{F}_2$-synthetic stable stems through 50

### 6.6. BP-synthetic stable stems through 50

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